DISCRIMINANTS: CALCULATION, PROPERTIES, AND CONNECTION TO THE ROOT DISTRIBUTION OF POLYNOMIALS WITH RATIONAL GENERATING FUNCTIONS

BY

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DISSERTATION

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Abstract

This research centers on discriminants and how discriminants and their q-analogues relate to the root distribution of polynomials. This topic includes the connections between the root distribution of a sequence of polynomials and the discriminant of the denominator of its generating function. A more specialized aspect of this area of this research concerns the root distribution and the generating functions of the discriminants of polynomials related to Chebyshev polynomials. Other areas involve the factorization properties and asymptotic behaviors of q-analogues of discriminants of cubic polynomials, and the diagonal sequence of the resultant of a certain pair of reciprocal polynomials.

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Contents

Chapter 1

Introduction

We study discriminants and connections between discriminants and root distribution of polynomials with rational generating functions. A familiar example of a discriminant is the discriminant of the quadratic polynomial $P(x) = ax^2 + bx + c$. Its discriminant is

$$\mathrm{Disc}_x(P(x)) = b^2 - 4ac$$

which equals zero if and only if $P(x)$ has a double root. This concept can be generalized to polynomials of higher degree with several variables. We shall focus on discriminants of polynomials in one variable. When polynomials of several variables arise, we consider the discriminant with respect to one variable at a time.

The study of discriminants has a long history, and includes contributions from both Stieltjes and Hilbert (see [1] and [6] respectively). Formulas for various specific types of polynomials are given, e.g., in [1], [2], [5], [6], and [7]. The concept of discriminant connects with the ratios of roots of polynomials. In particular the discriminant of a polynomial is zero if there is a double root, i.e., a pair of roots with ratio 1. Mourad Ismail [8] introduced the concept of generalized q-discriminant, which equals zero if there is a pair of roots with ratio q. Various formulas from q-series can be applied to compute q-discriminants of q-orthogonal polynomials. A precise definition for the discriminant of a one variable polynomial $P(x)$ of degree n with the leading coefficient p is

$$\mathrm{Disc}_x(P(x)) = p^{2n-2} \prod_{1 \le i < j \le n} (x_i - x_j)^2 \tag{1.1}$$

where $x_1, \ldots, x_n$ are roots of $P(x)$. The q-analogue of such a discriminant is defined by the formula:

$$\mathrm{Disc}_x(P;q) = p^{2n-2} q^{n(n-1)/2} \prod_{1\le i<j\le n} (q^{-1/2}x_i - q^{1/2}x_j)(q^{1/2}x_i - q^{-1/2}x_j). \tag{1.2}$$

We may refer to these as the ordinary and q-discriminant respectively. In the case $q = 1$, the latter reduces to the ordinary discriminant. The discriminant is a special case of the resultant between a polynomial and its derivative. The resultant between any two polynomials equals zero when these two polynomials share a common root. More precisely, the resultant of $P(x)$ and $R(x)$ is given by

$$\begin{aligned} \mathrm{Res}_x(P(x), R(x)) &= p^m \prod_{P(x_i)=0} R(x_i) \\ &= r^n \prod_{R(x_i)=0} P(x_i) \end{aligned}$$

where r and m are the leading coefficient and the degree of $R(x)$ respectively. The discriminant of $P(x)$ can be computed in terms of the resultant between it and its derivative as follow:

$$\begin{aligned} \mathrm{Disc}_x P(x) &= (-1)^{n(n-1)/2} \frac{1}{p} \mathrm{Res}(P(x), P'(x)) && (1.3) \\ &= (-1)^{n(n-1)/2} p^{n-2} \prod_{i\le n} P'(x_i). && (1.4) \end{aligned}$$

Our study will start with some new examples in which we compute the ordinary discriminants and q-discriminants of some simply defined polynomials. Even in these basic cases, some interesting properties (i.e. the connections between triangular numbers and Chebyshev polynomials, the iteration factorization of q-discriminants of cubic polynomials) emerge. We analyze the connections between discriminants and the root distribution of polynomials in Chapter 4. Here we show that the discriminants of certain polynomials related to Chebyshev polynomials factor into the product of two polynomials, one of which has coefficients that are much larger than the other's. Remarkably, these polynomials of dissimilar size have almost the same roots, and their discriminants involve exactly the same prime factors. The main goal of Chapter 5 and Chapter 6 is to analyze connections between the root distribution of polynomial sequences generated by rational functions and the discriminants of the polynomials that occur in the generating function. In several cases, the roots of these polynomials stay on the union of several fixed algebraic curves, whose endpoints are the roots of the discriminant of the denominator of the generating function. In Chapter 5, we start with a specific sequence of polynomials and determine the generating function. In Chapter 6, we start with a general generating function and show the roots of the generated polynomials lie on a union of fixed algebraic curves. For published versions of Chapter 4 and Chapter 5, see [3, 16, 17].

Chapter 2

New discriminant calculations: Triangular numbers and a diagonal sequence

In this chapter, we compute the resultants and discriminants of some self-reciprocal polynomials. A polynomial $p(z) = \sum_{i=0}^{n} a_i x^i$ is self-reciprocal if $a_i = \overline{a_{n-i}}$. A common approach to self-reciprocal polynomials is to let $z = x + x^{-1}$. With this substitution, we convert the original polynomial to a linear combination of Chebyshev polynomials. For examples of discriminants of various linear combinations of Chebyshev polynomials, see [5, 7]. Our first example gives a connection between triangular numbers and Chebyshev polynomials.

2.1 Triangular numbers and Chebyshev polynomials

The resultant of two polynomials, with arbitrary coefficients, even if both are self-reciprocal, is a rather complicated function of their coefficients. For example,

$$\begin{aligned} & \mathrm{Res}_x(1 + bx + cx^2 + bx^3 + x^4, 1 + sx + tx^2 + sx^3 + x^4) \\ = \; & (-2b^2 + c^2 + 4bs - bcs - 2s^2 + cs^2 + b^2t - 2ct - bst + t^2)^2. \end{aligned}$$

However, if most of the coefficients have a special form an elegant formula is a possibility. Here we consider a case in which only the middle coefficient is arbitrary and we obtain a closed form solution.

Proposition 2.1. *Let*

$$P_n(x, s) = \frac{x^{2n+3} - 1}{x - 1} - (2n + 2 - s)x^{n+1}$$

and

$$Q_n(x,t) = 1 + 3x + 6x^2 + \cdots + T_{n+1}x^n + T_n x^{n+1} + \cdots + 3x^{2n-1} + x^{2n} + tx^n$$

where T_n is the n-th triangular number. Then

$$\frac{s^2}{t^{2n+2}}\mathrm{Res}_x(P_n(x,s), Q_n(x,t)) = \left(U_{2n+2}\left(\frac{\sqrt{4+s/t}}{2}\right) - (2n+3) + s\right)^2$$

where $U_n(x)$ is the Chebyshev polynomial of the second kind.

Proof. Let

$$S_n(x) = x^n\left(\sum_{k=1}^{n} T_{n-k+1}(x^k + x^{-k}) + T_{n+1}\right).$$

Then the two polynomials $P_n(x,s)$ and $Q_n(x,t)$ can be written in terms of $S_n(x)$ as follows:

$$P_n(x,s) = (x-1)^2 S_n(x) + sx^{n+1}$$

and

$$Q_n(x,t) = S_n(x) + tx^n.$$

We note that if x is a root of $Q_n(x,t)$ then so is x^{-1} since $Q_n(x,t) = x^{2n}Q_n(x^{-1},t)$. The analogue of the latter equation for $P_n(x,s)$ is $P_n(x,s) = x^{2n+2}P(x^{-1},s)$. Denote the roots of $Q_n(x,t)$ by $x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}$. The definition of the resultant yields

$$\begin{aligned}
\mathrm{Res}_x(P_n(x,s), Q_n(x,t)) &= \prod_{i=1}^{n} P_n(x_i,s)P_n(x_i^{-1},s) \\
&= \prod_{i=1}^{n} x_i^{-2n-2}\prod_{i=1}^{n}(sx_i^{n+1} - tx_i^n(x_i-1)^2)^2 \\
&= t^{2n}\prod_{i=1}^{n}(\frac{s}{t} + 2 - x_i - x_i^{-1})^2.
\end{aligned}$$

Let

$$f_n(s,t) = \prod_{i=1}^{n} (\frac{s}{t} + 2 - x_i - x_i^{-1}).$$

It remains to prove that

$$U_{2n+2}\left(\frac{\sqrt{4+s/t}}{2}\right) - (2n+3) = \frac{s}{t} f_n(s,t) - s.$$

Denote the left side and right side of the equation above by $L_n(s,t)$ and $R_n(s,t)$ respectively. We will show that these two sequences of functions satisfy the same recurrence relation with the same initial condition. The recurrence relation for $L_n(s,t)$ can be deduced from the Chebyshev polynomial recurrence

$$(4x^2 - 2)U_n(x) = U_{n+2}(x) + U_{n-2}(x).$$

Replacing x by $\sqrt{4+s/t}/2$ and n by $2n+2$ yields

$$(2+s/t)U_{2n+2}\left(\frac{\sqrt{4+s/t}}{2}\right) = U_{2n+4}\left(\frac{\sqrt{4+s/t}}{2}\right) + U_{2n}\left(\frac{\sqrt{4+s/t}}{2}\right).$$

This gives us

$$(2+s/t)L_n(s,t) = L_{n+1}(s,t) + L_{n-1}(s,t) - (2n+3)s/t.$$

To find the recurrence relation for $R_n(s,t)$ we notice that $f_n(s,t)$ is the polynomial $Q_n(s,t)/x^n$ with $s/t + 2 = x + x^{-1}$. Since $(x+x^{-1})(x^k + x^{-k}) = (x^{k+1} + x^{-k-1}) + (x^{k-1} + x^{-k+1})$ for $1 \le k \le n$, we have

$$\begin{aligned}
(s/t+2)f_n(s,t) &= (x+x^{-1})\left(\sum_{k=1}^{n} T_{n-k+1}(x^k + x^{-k}) + T_{n+1} + t\right) \\
&= t(x+x^{-1}) + T_{n+1}(x+x^{-1}) \\
&\quad + \sum_{k=1}^{n} T_{n-k+1}(x^{k+1} + x^{-k-1}) + \sum_{k=1}^{n} T_{n-k+1}(x^{k-1} + x^{-k+1})
\end{aligned}$$

$$
\begin{aligned}
&= t(x+x^{-1}) + 2T_n \\
&\quad + \sum_{k=1}^{n+1} T_{n+2-k}(x^k + x^{-k}) + \sum_{k=1}^{n-1} T_{n-k}(x^k + x^{-k}) \\
&= f_{n+1} + f_{n-1} + t(x + x^{-1} - 2) - (2n+3) \\
&= f_{n+1} + f_{n-1} + s - (2n+3).
\end{aligned}
$$

By multiplying both sides by s/t and noting that $R_n(s,t) + s = sf_n(s,t)/t$, we obtain

$$(s/t+2)\left(R_n(s,t)+s\right) = R_{n+1}(s,t) + R_{n-1}(s,t) + 2s + (s-(2n+3))s/t.$$

After subtracting $(s/t+2)s$ from both sides, we see that the sequence $R_n(s,t)$ satisfies the same recurrence relation. Thus the proposition follows after checking initial values.

$\square$

2.2 A linear polynomial transformation and its root distribution

In this section, we consider a transformation L defined by

$$
\begin{cases}
L(z^m) = z^m, & \text{if } m \equiv 0, 3 \bmod 4, \\
L(z^m) = -z^m, & \text{if } m \equiv 1, 2 \bmod 4,
\end{cases}
$$

and extended to polynomials by linearity. We consider the polynomial $L((1+z)^m)$.

Proposition 2.2. *The polynomial $L((1+z)^m)$ has only real roots.*

Proof. By the definition of L, we note that

$$(1+iz)^m - i(1-iz)^m = (1-i)L((1+z)^m).$$

So the roots of $L((1+z)^m)$ satisfy

$$i = \left(\frac{1+iz}{1-iz}\right)^m. \tag{2.1}$$

The claim follows from the fact that the map

$$\frac{1+iz}{1-iz}$$

maps the real line to the unit circle.

$\square$

We will provide another proof of this proposition using the generating function of $L((1+z)^m)$ in Chapter 6. The next proposition shows that the polynomial $L((1+z)^m)$ arises from the resultant of two familar polynomials.

Proposition 2.3. *Let*

$$p_{2m}(x) = \sum_{k=0}^{2m} x^k - zx^m.$$

Then

$$\mathrm{Res}_x\left(p_{2m}(x), p_{2m-2}(x)\right) = \left(z^m L(1+1/z)^m\right)^2.$$

Proof. The definition of resultant gives

$$\begin{aligned}
\mathrm{Res}_x\left(p_{2m}(x), p_{2m-2}(x)\right) &= \prod_{p_{2m-2}(x)=0} p_{2m}(x) \\
&= \prod_{p_{2m-2}(x)=0} \left(\sum_{k=0}^{2m} x^k - x\sum_{k=0}^{2m-2} x^k\right) \\
&= \prod_{p_{2m-2}(x)=0} x^{2m}+1 \\
&= \prod_{x^{2m}+1=0} \frac{x^{2m-1}-1}{x-1} - zx^{m-1}
\end{aligned}$$

$$
\begin{aligned}
&= \prod_{x^{2m}+1=0} zx^{m-1} + \frac{x+1}{x(x-1)} \\
&= \prod_{x^{2m}+1=0} x^{m-1}\left(z + \frac{x+1}{x^m(x-1)}\right) \\
&= \prod_{x^m=i}\left(z + \frac{x+1}{i(x-1)}\right)^2,
\end{aligned}
$$

where the last equality comes from the fact that

$$
\frac{x+1}{x^m(x-1)} = \frac{x^{-1}+1}{x^{-m}(x^{-1}-1)}.
$$

Thus the roots of this resultant are

$$
z = -\frac{x+1}{i(x-1)}.
$$

This gives

$$
x = \frac{1+iz^{-1}}{1-iz^{-1}}.
$$

The claim follows from this identity and (2.1).

□

2.3 The diagonal sequence of a resultant

In this section, we consider the resultant in x of $S_n(x,s)$ and $S_{n+1}(x,t)$ where $S_n(x,s)$ is a self-reciprocal polynomial given by the formula

$$
S_n(x,s) = 1 + 2x + 3x^2 + \cdots + nx^{n-1} + (n+1)x^n + nx^{n+2} + \cdots + 2x^{2n-1} + x^{2n} + sx^n.
$$

If we view this polynomial as a polynomial of degree n in $z = x + x^{-1}$, the polynomial $S_n(z,s) = x^{-n}S_n(x,s)$ satisfies a nonhomogeneous recurrence relation

$$S_{n+1}(z,s) = zS_n(z,s) - S_{n-1}(z,s) - zs + 2s + 2.$$

The discriminants and resultants of orthogonal polynomials and their q-analogues satisfying three-terms homogeneous recurrence relations are given by [8].

Despite the simple definition of the $S_n(x,s)$, their resultants are interesting and nontrivial to determine. In the case $n = 6$ the resultant is the square of a polynomial in s, t whose coefficients are given by the following table:

	s^0	s^1	s^2	s^3	s^4	s^5	s^6	s^7
t^0	1	7	21	35	35	21	7	1
t^1	6	30	60	60	30	6	0	0
t^2	15	50	60	30	5	0	0	0
t^3	20	40	24	4	0	0	0	0
t^4	15	15	3	0	0	0	0	0
t^5	6	2	0	0	0	0	0	0
t^6	1	0	0	0	0	0	0	0
t^7	0	0	0	0	0	0	0	0

From this table, we see that the coefficient of s^at^b is 0 if $a+b > 7$, $b \geq 1$. Also, the first row and the first column are the coefficients from the expansions of $(s+1)^7$ and $(t+1)^6$ respectively. The proofs of these two facts are not difficult and are given in Proposition 2.5 and Proposition 2.4. The fact that the diagonal above the secondary diagonal is 1, 2, 3, 4,. . ., $n+1$ is harder, and the proof is given in Theorem 2.8. The first four rows of the table are generated by $(1+s)^7$, $6(1+s)^5$, $5(1+s)^3(3+s)$ and $4(1+s)(5+5s+s^2)$. The multiplicities at $s = -1$ of these polynomials are given in Theorem 2.9. Similar results hold for the first three columns of the table.

To establish these results, we first note that

$$S_n(x,s) = (x^n + x^{n-1} + \cdots + 1)^2 + sx^n.$$

Throughout the remainder of this section, we denote by $x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}$ the roots of $S_n(x,s)$. The definition of the resultant yields

$$\begin{aligned} \mathrm{Res}_x(S_n(x,s), S_{n+1}(x,t)) &= \prod_{i=1}^{n} S_{n+1}(x_i,t) S_{n+1}(x_i^{-1},t) \\ &= \prod_{i=1}^{n} x_i^{-2n-2} \prod_{i=1}^{n} S_{n+1}^2(x_i,t). \end{aligned}$$

Similarly,

$$\mathrm{Res}_x(S_n(x,s), S_{n+1}(x,t)) = \prod_{i=1}^{n+1} y_i^{-2n} \prod_{i=1}^{n+1} S_n^2(y_i,s),$$

where $y_1, y_1^{-1}, \ldots, y_{n+1}, y_{n+1}^{-1}$ are the roots of $S_{n+1}(y,t)$.

In particular, $\mathrm{Res}_x(S_n(x,s), S_{n+1}(x,t)) = f_n^2(s,t)$ where $f_n(s,t)$ is a polynomial in s and t given by

$$\begin{aligned} f_n(s,t) &= \prod_{i=1}^{n} \left(\frac{(x_i^{n+2}-1)^2}{(x_i-1)^2 x_i^{n+1}} + t \right) \\ &= \prod_{i=1}^{n+1} \left(\frac{(y_i^{n+1}-1)^2}{(y_i-1)^2 y_i^{n}} + s \right). \end{aligned}$$

.

Proposition 2.4. *With the definition of $f_n(s,t)$ as above, we have*

$$\begin{aligned} f_n(0,t) &= (t+1)^n, \\ f_n(s,0) &= (s+1)^{n+1}. \end{aligned}$$

Proof. Suppose $s = 0$. Then we have $x_i^n + \cdots + x_i + 1 = 0$. Hence

$$f_n(0,t) = \prod_{i=1}^{n} S_{n+1}(x_i, t) = \prod_{i=1}^{n} (t+1)^n = (t+1)^n.$$

Similarly, if $t = 0$ then $y_i^{n+2} = 1$ where y_i are roots of $S_{n+1}(x,t)$. Thus

$$f_n(s,0) = \prod_{i=1}^{n+1} S_n(y_i, s) = \prod_{i=1}^{n+1} (s+1)^n = (s+1)^{n+1}.$$

□

Proposition 2.5. *In the polynomial $f_n(s,t)$, the coefficient of $s^a t^b$ is 0 if $a + b > n$, $b \geq 1$.*

Proof. Let $\zeta_i = y_i + y_i^{-1}$ where y_i are roots of $S_{n+1}(x,t)$. We have

$$\begin{aligned} f_n(s,t) &= \prod_{i=1}^{n+1} (y_i^n + y_i^{-n} + 2(y_i^{n-1} + y_i^{-n+1}) + \cdots + n(y_i + y_i^{-1}) + n + 1 + s) \\ &= \prod_{i=1}^{n+1} (Q_n(\zeta_i) + s), \end{aligned}$$

where $Q_n(\zeta_i)$ is a polynomial of degree n in ζ_i. The coefficient of s^a is

$$\sum_{\substack{S \subset \{1,\ldots,n\} \\ |S| = n+1-a}} \prod_{i \in S} Q_n(\zeta_i).$$

We note that the summation above is a symmetric polynomial in $\zeta_1, \ldots, \zeta_{n+1}$ of degree at most $(n+1-a)n$, whereas, from the definition of $S_{n+1}(y,t)$, $t^b = t^b(\zeta_1, \ldots, \zeta_{n+1})$ is a symmetric polynomial in $\zeta_1, \ldots, \zeta_{n+1}$ of degree $b(n+1) > (n+1-a)n$. Thus the coefficient of $s^a t^b$ is 0.

□

We note that in $f_n(s,t)$ the coefficient of $s^a t^{n-a}$ is $a+1$ for any $0 \leq a \leq n$. To prove this, we first need two lemmas.

Lemma 2.6. *Suppose* $-n < a < n$ *and* $a \neq 0$. *Then*

$$\sum_{i=1}^{n} x_i^a = -1.$$

Proof. We note that x_i, $1 \leq i \leq n$, are the roots of

$$\frac{(x^n - 1)^2}{x - 1} - sx^n.$$

Since $-n < a < n$, we have

$$\sum_{i=1}^{n} x_i^a = \sum_{i=1}^{n} \omega_i^a = -1$$

where $\omega_i \neq 1$, $1 \leq i \leq n$, are the $(n+1)$-st roots of unity.

□

Lemma 2.7. *Suppose* $1 \leq a < n$. *The symmetric polynomial*

$$\sum_{i=1}^{n} \frac{(x_i^{n+2} - 1)^{2a}}{(x_i - 1)^{2a} x_i^{(n+1)a}}$$

is a polynomial in s *whose* s^a*-coefficient is* $2(-1)^{a-1}$.

Proof. The main idea of the proof is that

$$\frac{(x_i^{n+2} - 1)^{2a}}{(x_i - 1)^{2a} x_i^{(n+1)a}}$$

is a polynomial in x_i and x_i^{-1} whose highest and lowest powers of x_i are $\pm(n+1)a$. We note that only the terms with absolute powers greater than or equal to na affect the leading coefficient of s^a. We will represent

$$\frac{(x_i^{n+2} - 1)^{2a}}{(x_i - 1)^{2a} x_i^{(n+1)a}}$$

in two different ways, one of which represents the contributions from the terms with powers of

x_i greater than or equal to na (high power terms). The other way represents the contributions from the terms with power in x_i less than or equal to $-na$ (low power terms). The first representation is

$$\begin{aligned}
\sum_{i=1}^{n} \frac{(x_i^{n+2}-1)^{2a}}{(x_i-1)^a x_i^{(n+1)a}} &= \sum_{i=1}^{n} \frac{(x_i(1+x_i+x_i^2+\cdots+x_i^n)+1)^{2a}}{x_i^{(n+1)a}} \\
&= \sum_{i=1}^{n} \frac{(x_i^2(1+x_i+x_i^2+\cdots+x_i^n)^2+2x_i(1+x_i+x_i^2+\cdots+x_i^n)+1)^{a}}{x_i^{(n+1)a}}.
\end{aligned}$$

From this representation, the contribution to the coefficient of s^a from the high power terms only comes from

$$\sum_{i=1}^{n} \frac{x_i^{2a}(1+x_i+x_i^2+\cdots+x_i^n)^{2a}}{x_i^{(n+1)a}}.$$

Since

$$s = -\frac{(1+x+x_i^2+\cdots+x_i^n)^2}{x_i^n},$$

this contribution is

$$(-1)^a \sum_{i=1}^{n} x_i^a = (-1)^{a-1}.$$

The second representation is

$$\begin{aligned}
\sum_{i=1}^{n} \frac{(x_i^{n+2}-1)^{2a}}{(x_i-1)^{2a} x_i^{(n+1)a}} &= \sum_{i=1}^{n} \frac{(x_i^{n+1}+(1+x_i+x_i^2+\cdots+x_i^n))^{2a}}{x_i^{(n+1)a}} \\
&= \sum_{i=1}^{n} \frac{(x_i^{2n+2}+2x_i^{n+1}(1+x_i+x_i^2+\cdots+x_i^n)+(1+x_i+x_i^2+\cdots+x_i^n)^2)^{a}}{x_i^{(n+1)a}}.
\end{aligned}$$

The contribution to the coefficient of s^a from the low power terms only comes from

$$\sum_{i=1}^{n} \frac{(1+x_i+x_i^2+\cdots+x_i^n)^{2a}}{x^{(n+1)a}}.$$

This contribution is

$$(-1)^a \sum_{i=1}^{n} x_i^{-a} = (-1)^{a-1}.$$

So the s^a-coefficient of

$$\sum_{i=1}^{n} \frac{(x_i^{n+2}-1)^{2a}}{(x_i-1)^{2a} x_i^{(n+1)a}}$$

is

$$(-1)^{a-1} + (-1)^{a-1} = 2(-1)^{a-1}.$$

□

Theorem 2.8. *In $f_n(s,t)$ the coefficient of $s^a t^{n-a}$ is $a+1$ for any $0 \le a \le n$.*

Proof. From Proposition 2.4, it suffices to prove the theorem for $0 < a < n$. We recall that

$$f_n(s,t) = \prod_{i=1}^{n} \left(\frac{(x_i^{n+2}-1)^2}{(x_i-1)^2 x_i^{n+1}} + t \right).$$

We denote the a-th elementary symmetric polynomial and the a-th power sum symmetric polynomial in

$$\frac{(x_i^{n+2}-1)^2}{(x_i-1)^2 x_i^{n+1}}$$

by e_a and c_a respectively. We note that e_a and c_a are polynomials in s of degree a whose k-th coefficients are denoted by $c_k(e_a)$ and $c_k(p_a)$ respectively. The coefficient of t^{n-a} in $f_n(s,t)$ is given by e_a. We proceed by induction with the hypothesis that $c_k(e_k) = k+1$, $k < a$. Then Newton's identity yields

$$e_a = \frac{1}{a} \sum_{k=1}^{a} (-1)^{k-1} e_{a-k} p_k.$$

Since e_{a-k} and p_k are polynomials in s of degrees $a-k$ and k respectively, we have

$$\begin{aligned} a c_a(e_a) &= \sum_{k=1}^{a} (-1)^{k-1} c_{a-k}(e_{a-k}) c_k(p_k) \\ &= \sum_{k=1}^{a} (-1)^{k-1} (a-k+1) c_k(p_k) \end{aligned}$$

$$\begin{aligned} &= \sum_{k=1}^{a-1}(-1)^{k-1}(a-k)c_k(p_k) + \sum_{k=1}^{a}(-1)^{k-1}c_k(p_k) \\ &= (a-1)a + \sum_{k=1}^{a}(-1)^{k-1}c_k(p_k). \end{aligned}$$

Thus it remains to show

$$2a = \sum_{k=1}^{a}(-1)^{k-1}c_k(p_k)$$

or

$$2 = (-1)^{a-1}c_a(p_a).$$

The theorem follows from Lemma 2.7.

□

Theorem 2.9. *Suppose* $0 \le 2a < n+1$. *Then* $(s+1)^{(n+1)-2a}$ *divides the coefficient of* t^a *in* $f_n(s,t)$.

Proof. Let e_a^* and p_a^* be the a-th elementary symmetric polynomial and the a-th power sum symmetric polynomial in

$$\frac{(x_i-1)^2 x_i^{n+1}}{(x_i^{n+2}-1)^2}.$$

Since

$$f_n(s,t) = \prod_{i=1}^{n}\left(\frac{(x_i^{n+2}-1)^2}{(x_i-1)^2 x_i^{n+1}} + t\right),$$

the coefficient of t^a in $f_n(s,t)$ is

$$e_a^* \prod_{i=1}^{n}\frac{(x_i^{n+2}-1)^2}{(x_i-1)^2 x_i^{n+1}} = (s+1)^{n+1}e_a^*.$$

Thus it suffices to show that $(s+1)^{2a}e_a^*$ is a polynomial in s. We proceed by induction on a. The claim holds for $a=0$ by Proposition 2.4. From Newton's identity

$$e_a^* = \frac{1}{a}\sum_{k=1}^{a}(-1)^{k-1}e_{a-k}^* p_k^*$$

we have

$$(s+1)^{2a}e_a^* = \frac{1}{a}\sum_{k=1}^{a}(-1)^{k-1}\left((s+1)^{2(a-k)}e_{a-k}^*\right)\left((s+1)^{2k}p_k^*\right).$$

By the induction hypothesis, it suffices to show that $(s+1)^{2k}p_k^*$ is a polynomial in s. Since

$$s = -\frac{(1+x+x_i^2+\cdots+x_i^n)^2}{x_i^n},$$

we have

$$(s+1)^{2k}p_k^* = \sum_{i=1}^{n}\frac{1}{x_i^{k(n-1)}}\left(\frac{(x_i^{n+1}-1)^2+(x_i-1)^2x_i^n}{(x_i-1)(x_i^{n+2}-1)}\right)^{2k}.$$

Suppose ω is a $(n+2)$-nd root of unity. Then

$$(\omega^{n+1}-1)^2+(\omega-1)^2\omega^n = \left(\frac{1}{\omega}-1\right)^2-(\omega-1)^2\frac{1}{\omega^2}=0.$$

This implies that

$$\left(\frac{(x_i^{n+1}-1)^2+(x_i-1)^2x_i^n}{(x_i-1)(x_i^{n+2}-1)}\right)^{2k}$$

is a polynomial in x_i. Thus

$$\sum_{i=1}^{n}\frac{1}{x_i^{k(n-1)}}\left(\frac{(x_i^{n+1}-1)^2+(x_i-1)^2x_i^n}{(x_i-1)(x_i^{n+2}-1)}\right)^{2k}$$

is a polynomial in s. This completes the proof.

□

Corollary 2.10. *The coefficient of t in $f_n(s,t)$ is $n(s+1)^{n-1}$.*

Proof. This follows from Proposition 2.5, Theorem 2.9, and Theorem 2.8.

□

Remark: By an argument similar to the one in the proof of Theorem 2.9, one can show that if $0 \le 2a < n$, then $(t+1)^{n-2a}$ divides the coefficient of s^a in $f_n(s,t)$.

Chapter 3

A property of q-discriminants of certain cubics

In this chapter, we analyze the connections between the q-discriminants of certain cubic polynomials and the asymptotic behavior of a non-linear recurrence relation. The very existence of a solution to a differential equation (except in the linear case) is likely to depend sensitively on the initial conditions. Thus it is plausible by analogy that the asymptotic behavior of a non-linear recurrence may depend sensitively on the initial conditions.

We consider the recurrences

$$\vec{x}_{n+1} = (F_1(\vec{x}_n), F_2(\vec{x}_n), \ldots, F_m(\vec{x}_n))$$

where

$$\vec{x}_i = (x_{i1}, x_{i2}, \ldots, x_{im}) \in \mathbb{Q}^m,$$

and each F_j is a polynomial in m variables with rational coefficients. We write $\vec{x}_n \to \vec{x}_{n+1}$ or

$$\vec{x}_n \to (F_1(\vec{x}_n), \ldots, F_m(\vec{x}_n)).$$

We call these P-recurrences.

Example. *Consider the Diophantine Equation (D.E.)*

$$x + 2y = 1$$

and the recurrence

$$(x, y) \to (1 + 2(x + 2y - 1)^2, (x + 2y - 1)^2)$$

where $x, y \in \mathbb{Z}$. *Here the asymptotic behavior of its iterates depends upon whether or not the initial conditions satisfy the D.E. If (x,y) is a solution, then*

$$(x, y) \to (1, 0) \to (1, 0) \to \dots$$

is bounded. If (x, y) *is not a solution, the first iterate has the form* $(1 + 2t, t)$, $t \neq 0$, *and*

$$(1 + 2^m t^q, 2^{m-1} t^q) \to (1 + 2^{2m+3} t^{2q}, 2^{2m+2} t^{2q}).$$

So for $t \geq 1$ *the iterates are unbounded.*

An interesting question one can ask is this. Given a polynomial Diophantine equation, which P-recurrences have an asymptotic behavior that depends upon whether or not the initial condition satisfies the Diophantine equation? We limit our objective to the following Diophantine equation

$$y^2 = -27 - 4a^3 - 4b^3 + 18ab + a^2b^2 \tag{3.1}$$

with the P-recurrence $T^n(a, b, c, d)$ defined by

$$T : (a, b, c, d) \to \left(c, d, \frac{1}{2}(cd - 3 + a^3 - b^3), \frac{1}{2}(cd - 3 + b^3 - a^3)\right).$$

A special case of this P-recurrence is

$$T(a, a, a, a) = (a, a, \frac{1}{2}(a^2 - 3), \frac{1}{2}(a^2 - 3))$$

which reduces to a quadratic iteration

$$a \to \frac{1}{2}(a^2 - 3)$$

in each coordinate.

Definition 3.1. *The iteration sequence of* (a, b, c, d) *is*

$$(s_1, s_2, s_3, \ldots) = (a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2, a_3, \ldots)$$

where

$$\begin{aligned} T(a_i, b_i, c_i, d_i) &= (c_i, d_i, a_{i+1}, b_{i+1}) \\ T(c_i, d_i, a_{i+1}, b_{i+1}) &= (a_{i+1}, b_{i+1}, c_{i+1}, d_{i+1}) \end{aligned}$$

and

$$(a_1, b_1, c_1, d_1) = (a, b, c, d).$$

We are interested in the limit points of

$$\frac{\log |s_{i+1}|}{\log |s_i|}.$$

Since $T(a, a, a, a) = (a, a, (a^2 - 3)/2, (a^2 - 3)/2)$ we get at least two limit points 1 and 2.

Our (unattained) goal here is the loosely stated conjecture at the beginning of Section 3.2. We shall provide evidence for the conjecture by calculating many limit values in the case of factorization. Our work then suggests a further conjecture that is stated at the very end of this chapter.

We will now see how the Diophantine equation (3.1) and the P-recurrence defined above connect with the q-discriminant of a cubic polynomial. By decomposing the product in the definition of q-discriminant into elementary symmetric polynomials, we obtain the following explicit form of the q-discriminant.

Proposition 3.2. *The q-discriminant of the cubic polynomial $1 + ax + bx^2 + x^3$ is*

$$-(1+q+q^2)^3 + q(1+q+q^2)(1+4q+q^2)ab + q^3a^2b^2 - q(1+q)^2a^3 - q^2(1+q)^2b^3.$$

Throughout this section, we denote the q-discriminant above by $-Q(a,b,q)$.

3.1 Factorization of the q-Discriminant

We will consider the conditions under which $Q(a,b,q)$ factors over thte rationals. The factorization of $Q(a,b,q)$ depends sensitively on the values of a and b. Initial numerical experimentation first reveals four cases.

(1) $\mathrm{Disc}_q Q(a,b,q) = 0$. This corresponds to $a = b$ or $a = 1/b$.

(2) $a = -b-2$ or $a = -b-3$.

(3) $a = 3$, $b = -(m^2+3m+6)$.

(4) $a = m^2+5$, $b = (m+1)^2+5$.

Case (4) is of special interest, and leads to yet further cases of factorization. Let

$$\begin{aligned} \alpha(m) &= 17 + 2m + 2m^2 \\ \beta(m) &= 10 + 8m + 9m^2 + 2m^3 + m^4. \end{aligned}$$

Then

$$Q(m^2+5, (m+1)^2+5, q) = (1 - \alpha(m)q + \beta(m)q^2 - q^3)(1 - \beta(m)q + \alpha(m)q^2 - q^3)$$

and also

$$T(m^2+5,(m+1)^2+5,\alpha(m),\beta(m)) = (\alpha(m),\beta(m),\gamma(m),\delta(m))$$

where

$$\begin{aligned}
\gamma(m) &= 38-30m+42m^2-6m^3+12m^4+m^6,\\
\delta(m) &= 129+186m+147m^2+74m^3+27m^4+6m^5+m^6.
\end{aligned}$$

Now

$$Q(\alpha,\beta,q) = (1-\gamma q+\delta q^2-q^3)(1-\delta q+\gamma q^2-q^3).$$

We will study this factorization in more detail. We first need the following lemma.

Lemma 3.3. *If the quadruple of integers (a, b,c,d) satisfy the following equations*

$$\begin{cases} c+d &= ab-3,\\ cd &= 9-6ab+a^3+b^3, \end{cases} \tag{3.1}$$

then

$$Q(c,d,q) = (1-eq+fq^2-q^3)(1-fq+eq^2-q^3)$$

where e, f are the integers satisfying $T(a,b,c,d)=(c,d,e,f)$.

Proof. We first note that

$$e = \frac{1}{2}(cd-3+a^3+b^3) = 3-3ab+a^3$$

and

$$f = \frac{1}{2}(cd-3+b^3-a^3) = 3-3ab+b^3.$$

From (3.1), we have

$$c^3 + d^3 = (ab-3)^3 - 3(9 - 6ab + a^3 + b^3)(ab-3).$$

After substituting this into the formula for $Q(c, d, q)$ we obtain

$$\begin{aligned}
Q(c,d,q) &= (1+q+q^2)^3 - q(1+q+q^2)(1+4q+q^2)cd \\
&\quad -a^2b^2q^3 + q^2(1+q)^2(c^3+d^3) \\
&= (1+q+q^2)^3 - q(1+q+q^2)(1+4q+q^2)(ab-3) \\
&\quad -a^2b^2q^3 + q^2(1+q)^2((ab-3)^3 - 3(9-6ab+a^3+b^3)(ab-3)) \\
&= (1-(3-3ab+a^3)q + (3-3ab+b^3)q^2 - q^3) \\
&\quad (1-(3-3ab+b^3)q + (3-3ab+a^3) - q^3) \\
&= (1-eq+fq^2-q^3)(1-fq+eq^2-q^3).
\end{aligned}$$

We have proved the claim.

$\square$

From the lemma, we have the following theorem of factorization, assuming that the initial factorization occurs.

Theorem 3.4. *Suppose* $Q(a,b,q) = (1-cq+dq^2-q^3)(1-dq+cq^2-q^3)$ *and* $T(a,b,c,d) = (c,d,e,f)$ *then*

$$Q(c,d,q) = (1-eq+fq^2-q^3)(1-fq+eq^2-q^3)$$

and

$$Q(e,f,q) = (1-gq+hq^2-q^3)(1-hq+gq^2-q^3)$$

where g, h *satisfy* $T(c,d,e,f) = (e,f,g,h)$.

Proof. Recall the formula for $Q(a,b,q)$:

$$\begin{aligned} Q(a,b,q) &= (1+q+q^2)^3 - q(1+q+q^2)(1+4q+q^2)ab - a^2b^2q^3 + q^2(1+q)^2(a^3+b^3) \\ &= 1+(3-ab)q+(6+a^3-5ab+b^3)q^2+(7+2a^3-6ab-a^2b^2+2b^3)q^3 \\ &\quad +(6+a^3-5ab+b^3)q^4+(3-ab)q^5+q^6. \end{aligned}$$

By equating the q-coefficients of the equation

$$Q(a,b,q) = (1-cq+dq^2-q^3)(1-dq+cq^2-q^3),$$

we see that the tuple (a,b,c,d) satisfies (3.1). Thus by Lemma 3.3,

$$Q(c,d,q) = (1-eq+fq^2-q^3)(1-fq+eq^2-q^3).$$

From the definition of e and f we have

$$\begin{aligned} e+f &= \frac{1}{2}(cd-3+a^3-b^3)+\frac{1}{2}(cd-3-a^3+b^3) \\ &= cd-3 \end{aligned}$$

and

$$\begin{aligned} ef &= \frac{1}{4}((cd-3)^2-(a^3-b^3)^2) \\ &= \frac{1}{4}((cd-3)^2-(a^3+b^3)^2+4a^3b^3) \\ &= \frac{1}{4}((cd-3)^2-(cd-9+6(c+d+3))^2+4(c+d+3)^3) \\ &= 9-6cd+c^3+d^3. \end{aligned}$$

So the tuple (c,d,e, f) satisfies (3.1). The claim follows from Lemma 3.3.

□

Corollary 3.5. *Say there are integers a, b, y such that*

$$y^2 = -27 - 4a^3 - 4b^3 + 18ab + a^2b^2.$$

Set $c = (ab - 3 - y)/2$ and $d = (ab - 3 + y)/2$. Then $Q(a, b, q)$ and $Q(c, d, q)$ factor over $\mathbb{Z}$ and

$$T(a, b, c, d) = (c, d, 3 + a^3 - 3ab, 3 + b^3 - 3ab).$$

Proof. We first see that ab and y have the same parity. Thus c and d are integers. It is easy to check that the tuple $(a,\ b,\ c,\ d)$ satisfies (3.1). Thus the claim follows.

□

3.2 Sensitive Asymptotics

It seems that the asymptotic behavior of $T^n(a, b, c, d)$ depends upon whether or not a, b, c and d satisfy the conditions of the Corollary 3.5.

Conjecture. *Consider the limit points of $\log |s_{i+1}|/ \log |s_i|$ where $\{s_1, s_2, \ldots\}$ is an iteration sequence. For most integer quadruples $\{a, b, c, d\}$ the limit points are simply 1 and 2. However, if $Q(a, b, q)$ factors as*

$$(1 - cq + dq^2 - q^3)(1 - dq + cq^2 - q^3)$$

then there are limit points distinct from 1 and 2, and conversely.

In this section we only consider the P-recurrences whose initial conditions satisfy the conjecture above. We also exclude the trivial case $(a_n, b_n) = (-1, -1)$ and $(a_n, b_n) = (3, 3)$ for some n.

Lemma 3.6. *For every n, we have*

$$\begin{aligned} a_n^2 - 4b_n &\geq 0, \\ b_n^2 - 4a_n &\geq 0. \end{aligned}$$

Proof. We note that a_n, b_n satisfy the Diophantine equation

$$\begin{aligned} y^2 &= -27 - 4a_n^3 - 4b_n^3 + 18a_nb_n + a_n^2b_n^2 \\ &= -27 + 2a_nb_n + (a_n^2 - 4b_n)(b_n^2 - 4a_n). \end{aligned}$$

Suppose $4b_n > a_n^2 > 0$. If $b_n^2 < 4a_n$ then $a_n < 4$. Thus (a_n, b_n) could only be $(1,1)$, $(2,1)$, $(2,2)$, $(3,1)$, and $(3,2)$ which do not satisfy the Diophantine equation above. Hence $b_n^2 > 4a_n$. Thus $a_n > 0$ and

$$b_n^2 - 4a_n \leq (a_n^2 - 4b_n)(b_n^2 - 4a_n) < 2a_nb_n - 27.$$

Combining with the fact that $a_n < 4b_n$, this gives

$$b_n^2 - 4b_n\sqrt{b_n} - 8\sqrt{b_n} + 27 < 0.$$

Thus $b_n \leq 3$ and hence (a_n, b_n) could only be $(1,1), (1,2), (2,2), (1,3), (2,3)$ which do not satisfy the Diophantine equation. Hence $a_n^2 < 4b_n$ and the claim follows.

□

Lemma 3.7. *For every n, we have $a_na_1 > 0$ and $b_nb_1 > 0$.*

Proof. From Lemma 3.6 and induction, we have

$$\begin{aligned} a_{n+1}a_1 &= (a_n^3 - 3a_nb_n + 3)a_1 \\ &= a_na_1(a_n^2 - 3b_n) \\ &> 0. \end{aligned}$$

Similarly $b_n b_1 > 0$.

$\square$

Lemma 3.8. *Either $|b_n| > |a_n|$ or $|b_n| < |a_n|$ for every n large.*

Proof. Suppose $|b_n| > |a_n|$. Then

$$b_{n+1}^2 - a_{n+1}^2 = (b_n - a_n)(a_n^2 + a_n b_n + b_n^2)\left((a_n + b_n)(a_n^2 - a_n b_n + b_n^2) - 6a_n b_n + 6\right).$$

Thus it suffices to show

$$|a_n + b_n|(a_n^2 - a_n b_n + b_n^2) > 6|a_n b_n| + 6.$$

It is clear that for n big, $|a_n + b_n| > 12$, since

$$|a_n + b_n| = |a_{n-1} + b_{n-1}||a_{n-1}^2 - a_{n-1}b_{n-1} + b_{n-1}^2|.$$

Thus the inequality follows.

$\square$

Since we are working with limits, we can assume that Lemma 3.8 holds for every n.

Lemma 3.9. *The following inequalities hold for n large:*

$$\begin{aligned} 3|b_n| &< a_n^2, \\ 3|a_n| &< b_n^2. \end{aligned}$$

Proof. We will show $|b_n| < a_n^2$. We can assume $|b_n| > |a_n|$. If $b_n > 0$ then this is clear from Lemma 3.6 and Lemma 3.7. If $b_n < 0$ then

$$a_n^2 = (a_{n-1}^3 - 3a_{n-1}b_{n-1} + 3)^2$$

$$\begin{aligned} &> 9a_{n-1}^2 b_{n-1}^2 \\ &> 9^{n-1} a_1 b_1^2 b_2^2 \cdots b_{n-1}^2. \end{aligned}$$

Also

$$\begin{aligned} |b_n| &= |b_{n-1}^3 - 3a_{n-1}b_{n-1} + 3| \\ &< 5|b_{n-1}|^3 \\ &< 5^{n-1}|b_1|^3 {b_2}^2 {b_3}^2 \cdots b_{n-1}^2. \end{aligned}$$

Thus if n is so large that $(9/5)^{n-1}a_1^2 > 3|b_1|$, then $3|b_n| < a_n^2$. Similarly $3b_n^2 > |a_n|$.

$\square$

We can assume Lemma 3.9 holds for every n with $a = a_1$, $b = b_1$.

Theorem 3.10. *The sequence*

$$\frac{\log b_n}{\log a_n}$$

converges.

Proof. Consider the function

$$f(t) = \frac{\log |a_n^3 + t|}{\log |b_n^3 + t|}$$

with $|t| < \min\{|a|, |b|\}$. We have

$$f'(t) = \frac{\log |a_n^3 + t|}{b_n^3 + t} - \frac{\log |b_n^3 + t|}{a_n^3 + t} \neq 0.$$

Thus $f(t)$ is monotonic and the fact that $f(t)$ is increasing or decreasing depends on the signs of a_n, b_n and $|a_n| - |b_n|$. Since these signs do not depend on n by Lemma 3.7 , we have that

$$\frac{\log|b_{n+1}|}{\log|a_{n+1}|} = \frac{\log|b_n^3 - 3a_nb_n + 3|}{\log|a_n^3 - 3a_nb_n + 3|}$$

is monotonic. Also $\log|b_n|/\log|a_n|$ is bounded since from Lemma 3.9

$$\frac{\log|b_n|}{\log|a_n|} < \frac{\log a_n^2}{\log|a_n|} = 2$$

for n large. This completes the proof.

$\square$

Theorem 3.11. *Let*

$$\alpha = \lim_{n\to\infty} \frac{\log|b_n|}{\log|a_n|}.$$

The sequences $\log|c_n|/\log|b_n|$, $\log|d_n|/\log|c_n|$ *and* $\log|a_{n+1}|/\log|d_n|$ *converge and*

$$\lim_{n\to\infty} \frac{\log|c_n|}{\log|b_n|} = \begin{cases} 2 - \frac{1}{\alpha} & \textit{if } |b| > |a|, ab > 0 \\ 1 + \frac{1}{\alpha} & \textit{if } |b| > |a|, ab < 0 \\ \frac{2}{\alpha} - 1 & \textit{if } |b| < |a|, ab > 0 \\ 1 + \frac{1}{\alpha} & \textit{if } |b| < |a|, ab < 0 \end{cases}$$

$$\lim_{n\to\infty} \frac{\log|d_n|}{\log|c_n|} = \begin{cases} \frac{\alpha+1}{2\alpha-1} & \textit{if } |b| > |a|, ab > 0 \\ \frac{2\alpha-1}{\alpha+1} & \textit{if } |b| > |a|, ab < 0 \\ \frac{2-\alpha}{2+\alpha} & \textit{if } |b| < |a|, ab > 0 \\ \frac{3}{\alpha+1} - 1 & \textit{if } |b| < |a|, ab < 0 \end{cases}$$

$$\lim_{n\to\infty}\frac{\log|a_{n+1}|}{\log|d_n|} = \begin{cases} \frac{3}{\alpha+1} & \textit{if } |b|>|a|, ab>0 \\ \frac{3}{2\alpha-1} & \textit{if } |b|>|a|, ab<0 \\ \frac{3}{\alpha+1} & \textit{if } |b|<|a|, ab>0 \\ \frac{3}{2-\alpha} & \textit{if } |b|<|a|, ab<0. \end{cases}$$

Proof. From Lemma 3.8, 3.7 and 3.9, we have

$$\log|a_nb_n-3-\sqrt{a_n^2b_n^2-4a_n^3-4b_n^3+18a_nb_n-27}| \sim \begin{cases} \log|b_n^3|-\log|a_nb_n| & \text{if } |b|>|a|, ab>0 \\ \log|a_nb_n| & \text{if } |b|>|a|, ab<0 \\ \log|a_n^3|-\log|a_nb_n| & \text{if } |b|<|a|, ab>0 \\ \log|a_nb_n| & \text{if } |b|<|a|, ab<0 \end{cases}$$

$$\log|a_nb_n-3+\sqrt{a_n^2b_n^2-4a_n^3-4b_n^3+18a_nb_n-27}| \sim \begin{cases} \log|a_nb_n| & \text{if } |b|>|a|, ab>0 \\ \log|b_n^3|-\log|a_nb_n| & \text{if } |b|>|a|, ab<0 \\ \log|a_nb_n| & \text{if } |b|<|a|, ab>0 \\ \log|a_n^3|-\log|a_nb_n| & \text{if } |b|<|a|, ab<0. \end{cases}$$

Thus the claim follows by noting that $c_n = \left(a_nb_n-3-\sqrt{a_n^2b_n^2-4a_n^3-4b_n^3+18a_nb_n-27}\right)/2$ and $d_n = \left(a_nb_n-3+\sqrt{a_n^2b_n^2-4a_n^3-4b_n^3+18a_nb_n-27}\right)/2$.

□

Conjecture. *The set of limit points is countably infinite. The smallest limit point is largest for*

$$\{5,6,10,17,38,\ldots\}.$$

Also the smallest limit point of this sequence is $\gamma = 1.2632474\ldots$.

For the sequence $\{5,6,10,17,38,\ldots\}$ in the conjecture, we can calculate any limit point to any desired accuracy. From the proof of Theorem 3.10, the sequence $\log b_n / \log a_n$ increases and converges. Denote by α the limit of this sequence. From Lemma 3.6, we have for n large,

$$\begin{aligned}
\frac{\log b_{n+1}}{\log a_{n+1}} &= \frac{\log(b_n^3 - 3a_nb_n + 3)}{\log(a_n^3 - 3a_nb_n + 3)} \\
&\leq \frac{3\log b_n}{3\log a_n - \log 4} \\
&\leq \frac{\log b_n}{\log a_n}\left(1 + \frac{\log 4}{2\log a_n}\right) \\
&\leq \frac{\log b_n}{\log a_n} + \frac{2\log 2}{\log a_n}.
\end{aligned}$$

Thus we can choose N large so that

$$\frac{\log b_N}{\log a_N} \leq \alpha \leq \frac{\log b_N}{\log a_N} + 2\log 2 \sum_{n=N}^{\infty} \frac{1}{\log a_n}.$$

Since $1/\log a_{n+1} \leq 1/(3\log a_n)$, we have

$$\frac{\log b_N}{\log a_N} \leq \alpha \leq \frac{\log b_N}{\log a_N} + \frac{3\log 2}{\log a_N}.$$

When $N = 10$, these inequalities imply $1.34785 < \alpha < 1.34786$. We can compute the remaining three limit points by Theorem 3.11.

We end this chapter by another conjecture.

Conjecture. *The infimum of all the limit points is* $1/2$*.*

For example, the smallest limit point of the sequence

$$\{149, -152, -22654, 3, 3375896, \ldots\}$$

is 0.50099380165.

Chapter 4

Factorization of discriminants of transformed Chebyshev polynomials: The Mutt and Jeff syndrome

In this chapter we look at certain integral transforms of Chebyshev polynomials. We show that their discriminants have remarkable properties. The integral transform is

$$I_x p(z) = \frac{1}{2}\int_{-z}^{z} p'(t)K(x,t)dt$$

where p is some Chebyshev polynomial. If $K(x,t) \equiv 1$, then $I_x U_n(z)$ is 0 or $U_n(z)$ depending upon whether n is even or odd. We show that in the special case of the "parabolic kernel" $K(x,t) = x - t^2$ the resulting polynomials have discriminants that factor in a remarkable way. More precisely, when $p = U_{2n-1}(z)$ the discriminant factors into two polynomials, one of which has coefficients that are much larger than the coefficients of the other. Moreover, these two polynomials have "almost" the same roots, and their discriminants involve exactly the same prime divisors. For example, when $n = 6$, these two polynomials are

$$\begin{aligned} M(x) &= -143 + 2002x - 9152x^2 + 18304x^3 - 16640x^4 + 5632x^5 \\ J(x) &= -2606483707 + 826014609706x - 10410224034496x^2 \\ &\quad +40393170792832x^3 - 60482893968640x^4 + 30616119778816x^5. \end{aligned}$$

The discriminants of $M(x)$ and $J(x)$ are $2^{64}3^4 11^3 13^4$ and $2^{40}3^4 11^{35} 13^{44}$ respectively. The roots of $M(x)$ rounded to 5 digits are

$$\{0.13438, 0.36174, 0.62420, 0.85150, 0.98272\}$$

whereas those of $J(x)$ are

$$\{0.0032902, 0.13452, 0.36181, 0.62428, 0.85163\}.$$

We notice that in this case, the discriminants of these two polynomials have the same (rather small) prime factors. And except for the root 0.98272 of $M(x)$ and 0.0032902 of $J(x)$, the remaining roots are pairwise close. In fact, for n large we can show, after deleting one root from each of $M(x)$ and $J(x)$, that the remaining roots can be paired in such a way that the distance between any two in a given pair is at most $1/2n^2$.

We call the small polynomial $M(x)$ the "Mutt" polynomial and the large polynomial $J(x)$ the "Jeff" polynomial after two American comic strip characters drawn by Bud Fisher. They were an inseparable pair, one of whom ("Mutt") was very short compared to the other ("Jeff"). These two names are suggested by Kenneth Stolarsky.

Now, what will happen if we take the discriminant in t of $U'_{2n-1}(t)(x-t^2)$ instead of taking the integral? It is not difficult to show that this discriminant is

$$Cx\left(U'_{2n-1}(\sqrt{x})\right)^4.$$

where C is a constant depending on n. As we will see, one remarkable property about the polynomial $U'_{2n-1}(\sqrt{x})$ is that its discriminant has the same prime divisors as the discriminants of our polynomials $M(x)$ and $J(x)$ and its roots are almost the same as those of these two polynomials.

The main results are given by Theorems 4.1, 4.2 and 4.3. In Section 4.1 we provide the reader with a convenient summary of notations and properties of Chebyshev polynomials, discriminants, and resultants. Section 4.2 defines the Mutt and the Jeff polynomials. Sections 4.3 and 4.4 analyze the discriminants of $J(x)$ and $M(x)$. Section 4.5 shows that the roots of these two polynomials are pairwise close after deleting one root from each.

4.1 Discriminant, resultant and Chebyshev polynomials

Here we compute the discriminants and resultants of some kinds of Chebyshev polynomials. Before these computations, we review some basic properties of the Chebyshev polynomials. The discriminants of the Chebyshev polynomials $T_n(x)$ and $U_n(x)$ are given by simple and elegant formulas (see [11] or [15]):

$$\mathrm{Disc}_x T_n(x) = 2^{(n-1)^2} n^n \tag{4.1}$$

and

$$\mathrm{Disc}_x U_n(x) = 2^{n^2} (n+1)^{n-2}. \tag{4.2}$$

Is there anything comparable for linear combinations or integral transforms of Chebyshev polynomials? For a special type of linear combination, formula (4.2) was generalized in [5] to

$$\mathrm{Disc}_x(U_n(x) + kU_{n-1}(x)) = 2^{n(n-1)} a_{n-1}(k),$$

where

$$a_{n-1}(k) = (-1)^n \frac{(2n+1)^n k^n}{(n+1)^2 - n^2k^2} \left(U_n \left(-\frac{a+1+nk^2}{(2n+1)k} \right) + kU_{n-1} \left(-\frac{n+1+nk^2}{(2n+1)k} \right) \right).$$

For various formulas related to this type of linear combination, see [5] and [7]. The Chebyshev polynomial of the second kind $U_n(x)$ has derivative given by the formula

$$\frac{dU_n(x)}{dx} = \frac{(n+1)T_{n+1} - xU_n}{x^2 - 1}. \tag{4.3}$$

The derivative of the Chebyshev polynomial of the first kind $T_n(x)$ is

$$T_n'(x) = nU_{n-1}(x). \tag{4.4}$$

The connections between the two Chebyshev polynomials are given by

$$\begin{aligned} T_n(x) &= \frac{1}{2}\left(U_n(x) - U_{n-2}(x)\right) && (4.5)\\ &= U_n(x) - xU_{n-1}(x) && (4.6)\\ &= xT_{n-1}(x) - (1-x^2)U_{n-2}(x). && (4.7) \end{aligned}$$

Mourad Ismail [8] applied the following theorem to compute the generalized discriminant of the generalized orthogonal polynomials.

THEOREM (Schur [12])

Let $p_n(x)$ be a sequence of polynomials satisfying the recurrence relation

$$p_n(x) = (a_n x + b_n) - c_n p_{n-2}(x)$$

and the initial conditions $p_0(x) = 1$, $p_1(x) = a_1 x + b_1$. Assume that $a_1 a_n c_n \neq 0$ for $n > 1$. Then

$$\prod_{p_n(x_i)=0} p_{n-1}(x_i) = (-1)^{n(n-1)/2} \prod_{j=1}^{n} a_j^{n-2j+1} c_j^{j-1}$$

for $n \geq 1$.

Ismail's idea is that if we can construct $A_n(x)$ and $B_n(x)$ so that

$$p_n'(x) = A_n(x)p_{n-1}(x) + B_n(x)p_n(x),$$

then

$$\mathrm{Disc}_x p_n(x) = \gamma^{n-2} \prod_{j=1}^{n} a_j^{n-2j+1} c_j^{j-1} \prod_{p_n(x_i)=0} A_n(x_i).$$

In the special case of the Chebyshev polynomial satisfying the recurrence relation $U_n(x) =$

$2xU_{n-1}(x) - U_{n-2}(x)$ and the differental equation

$$U_n'(x) \quad = \quad \frac{2xnU_n - (n+1)U_{n-1}}{x^2 - 1},$$

we obtain

$$\prod_{U_{2n}(x)=0} U_{2n-1}(x) = 1 \tag{4.8}$$

and

$$\mathrm{Disc}_x U_n(x) = 2^{n^2}(n+1)^{n-2}.$$

4.2 The Mutt and Jeff polynomial pair

In this section we will show that the discriminant of the integral transform of the derivative of the Chebyshev polynomial factors into the square of the product of the Mutt and Jeff polynomials whose formulas will be provided. While the formula for the Mutt polynomial can be given explicitly in terms of the Chebyshev polynomials, we can only describe the Jeff polynomial by its roots. By an integration by parts, we have

$$\begin{aligned} I_x U_{2n-1}(z) &= \frac{1}{2}\int_{-z}^{z} U_{2n-1}'(t)(x - t^2)dt \\ &= (x - z^2)U_{2n-1}(z) + \int_{-z}^{z} U_{2n-1}(t)tdt \\ &= (x - z^2)U_{2n-1}(z) + \frac{1}{2}\int_{-z}^{z} U_{2n}(t) + U_{2n-2}(t)dt \\ &= (x - z^2)U_{2n-1}(z) + \left(\frac{T_{2n+1}(z)}{2n+1} + \frac{T_{2n-1}(z)}{2n-1}\right). \end{aligned} \tag{4.1}$$

This is a polynomial in z with the leading coefficient

$$-2^{2n-1} + \frac{2^{2n}}{2n+1} = -\frac{(2n-1)}{(2n+1)}2^{2n-1}.$$

The discriminant of this polynomial in z is

$$
\begin{aligned}
& C_n \mathrm{Res}_z(I_x U_{2n-1}(z), U'_{2n-1}(z)(x-z^2)) \\
= \;& C_n \mathrm{Res}_z(I_x U_{2n-1}(z), x-z^2)\mathrm{Res}_z(I_x U_{2n}(z), U'_{2n-1}(z)) \\
= \;& C_n \left((2n-1)T_{2n+1}(\sqrt{x}) + (2n+1)T_{2n-1}(\sqrt{x})\right)^2 \left(\prod_{\substack{U'_{2n-1}(\zeta_i)=0 \\ \zeta_i>0}} I_x U_{2n-1}(\zeta_i) \right)^2
\end{aligned}
$$

where C_n is a rational number depending only on n and can be different in each occurrence. Also the factors of its numerator and denominator can only be 2 or factors of $2n-1$, $2n+1$. From this, we define the Mutt polynomial

$$
M(x) \;=\; \frac{(2n-1)T_{2n+1}(\sqrt{x}) + (2n+1)T_{2n-1}(\sqrt{x})}{x\sqrt{x}}.
$$

Also, we can define, within a plus or minus sign, the Jeff polynomial $J(x) \in \mathbb{Z}[X]$ as the polynomial of degree $n-1$ whose coefficients are relatively prime and for which

$$
C_n \left(\prod_{\substack{U'_{2n-1}(\zeta_i)=0 \\ \zeta_i>0}} I_x U_{2n-1}(\zeta_i) \right)^2 = A_n J^2(x), \tag{4.2}
$$

where A_n is a suitable rational number.

4.3 The discriminant of $J(x)$

In this section, we find the factors of the discriminant of $J(x)$. In particular we will prove the theorem:

Theorem 4.1. *The discriminant of $J(x)$ has the same prime factors as those of the dis-*

criminant of $U'_{2n-1}(\sqrt{x})$. *Also*

$$\mathrm{Disc}_x U'_{2n-1}(\sqrt{x}) = 3(2n+1)^{n-2}(2n-1)^{n-3}n^{n-3}2^{2n^2-3n-1}.$$

Since we do not have an explicit formula for $J(x)$, we will compute the discriminant as the square of the product of the distances between the roots. We first note that

$$\left(\prod_{\substack{U'_{2n-1}(\zeta_i)=0 \\ \zeta_i>0}} I_x U_{2n-1}(\zeta_i) \right)^2$$

is a polynomial in x whose leading coefficient is

$$\begin{aligned} \pm \prod_{U'_{2n-1}(\zeta_i)=0} U_{2n-1}(\zeta_i) &= \pm \frac{\mathrm{Disc}_x U_{2n-1}(x)}{(2n-1)^{2n-1}} \\ &= \pm \frac{2^{(2n-1)^2}(2n)^{2n-3}}{(2n-1)^{2n-1}}. \end{aligned}$$

Thus from (4.2), the factors of the leading coefficient of $J(x)$ can only be 2 or factors of $(2n-1)$, $(2n+1)$, n.

We now consider the roots of $J(x)$. According to the formula (4.1), $J(x)$ has $n-1$ real roots given by the formula

$$-\frac{T_{2n+1}(\zeta_i)}{(2n+1)U_{2n-1}(\zeta_i)} - \frac{T_{2n-1}(\zeta_i)}{(2n-1)U_{2n-1}(\zeta_i)} + \zeta_i^2$$

where $U'_{2n-1}(\zeta_i) = 0$ and $\zeta_i > 0$. The derivative formula (4.1) implies that $2nT_{2n}(\zeta_i) = \zeta_i U_{2n-1}(\zeta_i)$. Thus (4.7) gives

$$\begin{aligned} T_{2n+1}(\zeta_i) &= \zeta_i T_{2n}(\zeta_i) - (1-\zeta_i^2)U_{2n-1}(\zeta_i) \\ &= U_{2n-1}(\zeta_i)\left(\zeta_i^2(1+\frac{1}{2n}) - 1\right). \end{aligned}$$

Hence the roots of $J(x)$ can be written as below:

$$\begin{aligned}
& -\frac{T_{2n+1}(\zeta_i)}{(2n+1)U_{2n-1}(\zeta_i)} - \frac{T_{2n-1}(\zeta_i)}{(2n-1)U_{2n-1}(\zeta_i)} + \zeta_i^2 \\
= & \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)\frac{T_{2n+1}(\zeta_i)}{U_{2n-1}(\zeta_i)} - \frac{2\zeta_i T_{2n}(\zeta_i)}{(2n-1)U_{2n-1}(\zeta_i)} + \zeta_i^2 \\
= & \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)\left(\zeta_i^2\left(1+\frac{1}{2n}\right)-1\right) - \frac{\zeta_i^2}{n(2n-1)} + \zeta_i^2 \\
= & \ \zeta_i^2 - \frac{2}{(2n+1)(2n-1)}. \qquad (4.1)
\end{aligned}$$

Thus from the definition of discriminant as the product of differences between roots, it suffices to consider the discriminant of $U'_{2n-1}(\sqrt{x})$. The formulas (4.3) and (4.4) give

$$\begin{aligned}
2\sqrt{x}U''_{2n-1}(\sqrt{x}) &= -\frac{2\sqrt{x}}{x-1}U'_{2n-1}(\sqrt{x}) + \frac{1}{x-1}\left(4n^2U_{2n-1}(\sqrt{x}) - \sqrt{x}U'_{2n-1}(\sqrt{x}) - U_{2n-1}(\sqrt{x})\right) \\
&= \frac{-3\sqrt{x}}{x-1}U'_{2n-1}(\sqrt{x}) + \frac{4n^2-1}{x-1}U_{2n-1}(\sqrt{x}). \qquad (4.2)
\end{aligned}$$

We note that $U'_{2n-1}(\sqrt{x})$ is a polynomial of degree $n-1$ with the leading coefficient

$$(2n-1)2^{2n-1}.$$

From the definition of discriminant in terms of resultant (1.4) and the formulas (4.2) and (4.2), we have

$$\begin{aligned}
\mathrm{Disc}_x U'_{2n-1}(\sqrt{x}) &= (2n-1)^{n-3}2^{(2n-1)(n-3)} \prod_{U'_{2n-1}(\sqrt{x_i})=0} \frac{4n^2-1}{x_i-1}\frac{1}{2\sqrt{x}}U_{2n-1}(\sqrt{x}) \\
&= (2n-1)^{n-3}2^{(2n-1)(n-3)} \prod_{\substack{U'_{2n-1}(x_i)=0 \\ x_i>0}} \frac{4n^2-1}{2x_i(x_i^2-1)}U_{2n-1}(x_i) \\
&= \frac{(4n^2-1)^{n-1}}{2^{n-1}\sqrt{2n/(2n-1)2^{2n-1}U'_{2n-1}(1)/(2n-1)2^{2n-1}}}\sqrt{\mathrm{Disc}_x U_{2n-1}(x)}
\end{aligned}$$

$$
\begin{aligned}
& \times \frac{(2n-1)^{n-3}2^{(2n-1)(n-3)}}{\sqrt{(2n-1)^{2n-1}2^{(2n-1)(2n-2)}}} \\
= & \frac{(4n^2-1)^{n-1}}{2^{n-1}\sqrt{2n/(2n-1)2^{2n-1}}U'_{2n-1}(1)/(2n-1)2^{2n-1}} \\
& \times 2^{(2n-1)^2/2}(2n)^{(2n-3)/2}\frac{\sqrt{2n-1}}{(2n-1)^3 2^{2(2n-1)}} \\
= & (2n-1)^{n-2}(2n+1)^{n-1}n^{n-2}2^{n(2n-3)}/U'_{2n-1}(1),
\end{aligned}
$$

where $U'_{2n-1}(1)$ can be computed from its trigonometric definition:

$$
\begin{aligned}
U'_{2n-1}(1) &= -\lim_{\theta\to 0}\frac{2n\cos 2n\theta\sin\theta - \sin 2n\theta\cos\theta}{\sin^3\theta} \\
&= -\lim_{\theta\to 0}\frac{2n(1-2n^2\theta^2)(\theta-\theta^3/6)-(2n\theta-8n^3\theta^3/6)(1-\theta^2/2)}{\theta^3} \\
&= -\frac{2}{3}n(4n^2-1).
\end{aligned}
$$

Thus

$$
\mathrm{Disc}_x U'_{2n-1}(\sqrt{x}) = 3(2n+1)^{n-2}(2n-1)^{n-3}n^{n-3}2^{2n^2-3n-1}.
$$

Thus $\mathrm{Disc}_x J(x)$ has factors 2, 3, and factors of powers of n, $2n-1$, $2n+1$.

4.4 The discriminant of $M(x)$

In this section, we will present an explicit formula for the discriminant of $M(x)$ and show that this discriminant also has factors 2, 3, and factors of powers of n, $2n-1$, $2n+1$. In particular, we will prove the theorem:

Theorem 4.2. *The discriminant of $M(x)$ is*

$$
\mathrm{Disc}_x M(x) = \pm(2n-1)^{n-3}(2n+1)^{n-2}2^{2n^2-n-5}3n^{n-3}.
$$

We note that $M(x)$ is a polynomial in x of degree $n-1$ whose leading coefficient is

$(2n-1)2^{2n}$. To compute $M'(x)$ we notice the following fact:

$$(2n-1)T_{2n+1}(x) + (2n+1)T_{2n-1}(x) = 2(2n-1)(2n+1)\int_0^x tU_{2n-1}(t)dt. \tag{4.1}$$

From this, we have

$$M'(x) = -\frac{3}{2}\frac{M(x)}{x} + \frac{(2n+1)(2n-1)}{x\sqrt{x}}U_{2n-1}(\sqrt{x}).$$

Thus the definition of discriminant (1.4) yields

$$\begin{aligned}\mathrm{Disc}_x(M(x)) &= \pm(2n-1)^{n-3}2^{2n(n-3)} \\ &\quad \times(2n-1)^{n-1}(2n+1)^{n-1}\frac{(2n-1)2^{2n}}{M(0)}\prod_{M(x_i)=0} x_i^{-1/2}U_{2n-1}(\sqrt{x_i}),\end{aligned}$$

where the value of the constant coefficient of $M(x)$ can be obtained from (4.1)

$$M(0) = \pm 4(2n-1)(2n+1)n/3.$$

By substituting this value into the equation above, we obtain

$$\mathrm{Disc}_x M(x) = \frac{\pm 2^{2n^2-4n-2}(2n-1)^{2n-4}(2n+1)^{n-2}3}{n}\prod_{M(x_i)=0} x_i^{-1/2}U_{2n-1}(\sqrt{x_i}).$$

To compute the product above, we follow an idea from Gishe and Ismail [7] by writing $M(x)$ as a linear combination of $U_{2n-1}(x)$ and $U_{2n}(x)$. In particular this combination can be obtained from (4.5) and (4.6) as follows:

$$\begin{aligned}(2n-1)T_{2n+1}(\sqrt{x}) + (2n+1)T_{2n-1}(\sqrt{x}) &= -2T_{2n+1}(\sqrt{x}) + 2\sqrt{x}(2n+1)T_{2n}(\sqrt{x}) \\ &= -U_{2n+1}(\sqrt{x}) + U_{2n-1}(\sqrt{x}) \\ &\quad +2\sqrt{x}(2n+1)(U_{2n}(\sqrt{x}) - \sqrt{x}U_{2n-1}(\sqrt{x}))\end{aligned}$$

$$= \quad 4n\sqrt{x}U_{2n}(\sqrt{x}) - (2x(2n+1)-2)U_{2n-1}(\sqrt{x}).$$

Thus

$$\begin{aligned}
\prod_{M(x_i)=0} x^{-1/2}U_{2n-1}(\sqrt{x}) &= \frac{2^{(2n-1)(n-1)}}{(2n-1)^{n-1}2^{2n(n-1)}} \prod_{x_i^{-1/2}U_{2n-1}(\sqrt{x_i})=0} M(x_i) \\
&= \frac{2^{(2n-1)(n-1)}}{(2n-1)^{n-1}2^{2n(n-1)}} 4^{n-1}n^{n-1} \\
&\quad \times \frac{2^{2n-1}}{2n} \prod_{x^{-1/2}U_{2n-1}(\sqrt{x_i})=0} U_{2n}(\sqrt{x}) \\
&= \frac{n^{n-2}2^{3n-3}}{(2n-1)^{n-1}} \prod_{\substack{U_{2n-1}(x_i)=0 \\ x_i>0}} U_{2n}(x_i) \\
&= \pm\frac{n^{n-2}2^{3n-3}}{(2n-1)^{n-1}}
\end{aligned}$$

where the last equality is obtained from (4.8). Hence

$$\mathrm{Disc}_x M(x) = \pm(2n-1)^{n-3}(2n+1)^{n-2}2^{2n^2-n-5}3n^{n-3}.$$

From this we conclude that the discriminants of $M(x)$ and $J(x)$ have the same factors 2, 3, and all the factors of n, $2n-1$, $2n+1$.

4.5 The roots of $M(x)$ and $J(x)$

In this section we will show that the roots of $M(x)$ and $J(x)$ are pairwise similar except for one pair. In particular, we will prove the theorem:

Theorem 4.3. *For every root x_0 of $J(x)$ except the smallest root, there is a root of $M(x)$ in the interval*

$$[x_0 - 3/(10n^2), x_0 + 1/(2n^2)).$$

We first show that with one exception, the roots of $M(x)$ and $U'_{2n-1}(\sqrt{x})$ are pairwise close. From the definition, the roots of these two polynomial are positive real numbers. To simplify the computations, we consider $U'_{2n-1}(x)$ and the polynomial

$$R(x) = (2n-1)T_{2n+1}(x) + (2n+1)T_{2n-1}(x)$$

whose roots (except 0) are the square roots of the positive roots of $J(x)$ and $M(x)$ respectively. And it will suffice to consider the positive roots of $R(x)$ and $U'_{2n-1}(x)$.

Let ζ be a positive root of $U'_{2n-1}(x)$. We will show that for a certain small value $\delta > 0$, the quantities $R(\zeta)$ and $R(\zeta - \delta)$ have different signs and thus $R(x)$ admits a root in the small interval $(\zeta - \delta, \zeta)$. First, we observe that the equation (4.1) gives

$$R(\zeta) = 2U_{2n-1}(\zeta).$$

For some $t \in (\zeta - \delta, \zeta)$, we have

$$R(\zeta - \delta) = R(\zeta) - R'(t)\delta.$$

where

$$R'(t) = 2(2n+1)(2n-1)tU_{2n-1}(t).$$

Thus

$$R(\zeta - \delta) = 2U_{2n-1}(\zeta) - 2(2n+1)(2n-1)tU_{2n-1}(t)\delta. \tag{4.1}$$

To prove $R(\zeta - \delta)$ and $R(\zeta) = 2U_{2n-1}(\zeta)$ have different signs, it remains to show that $2(2n+1)(2n-1)tU_{2n-1}(t)\delta$ is sufficiently large in magnitude and has the same sign as $R(\zeta)$. To ensure this fact, we first impose the following two conditions on δ:

$$\begin{aligned} \zeta\delta &> \frac{A}{(2n+1)(2n-1)}, && (4.2) \\ \zeta &> 6\delta, && (4.3) \end{aligned}$$

where the value of A and the existence of δ with respect to these conditions will be determined later. With these conditions, we obtain the following lower bound for $R'(t)\delta$:

$$\begin{aligned} |2(2n+1)(2n-1)U_{2n-1}(t)t\delta| &> 2(2n+1)(2n-1)|U_{2n-1}(t)|(\zeta-\delta)\delta \\ &> \frac{5}{3}(2n+1)(2n-1)|U_{2n-1}(t)|\zeta\delta \\ &> \frac{5A}{3}|U_{2n-1}(t)|. && (4.4) \end{aligned}$$

We now need to show that $U_{2n-1}(t)$ is not too small compared to $U_{2n-1}(\zeta)$. A Taylor series expansion gives

$$U_{2n-1}(t) = U_{2n-1}(\zeta) + U''_{2n-1}(\zeta-\epsilon)(t-\zeta)^2 \tag{4.5}$$

where $0 < \epsilon < \zeta - t$. From (4.3), we have, for $-1 < x < 1$, the following trivial bound for $U'_n(x)$:

$$|U'_n(x)| < \frac{2n}{1-x^2}. \tag{4.6}$$

Thus (4.2), with $\sqrt{x}$ replaced by x, implies that for large n

$$\begin{aligned} |U''_{2n-1}(\zeta-\epsilon)| &\sim \frac{4n^2}{1-(\zeta-\epsilon)^2}|U_{2n-1}(\zeta-\epsilon)| && (4.7) \\ &< \frac{4n^2}{1-\zeta^2}|U_{2n-1}(\zeta)|. \end{aligned}$$

We impose another condition on δ:

$$\delta < \frac{\sqrt{1-\zeta^2}}{4n}. \tag{4.8}$$

With this condition, we have

$$|U''_{2n-1}(\zeta - \epsilon)(t-\zeta)^2| \;\; < \;\; \frac{1}{4}|U_{2n-1}(\zeta)|.$$

Now (4.5) implies that $U_{2n-1}(t)$ and $U_{2n-1}(\zeta)$ have the same sign and

$$|U_{2n-1}(t)| > \frac{3}{4}|U_{2n-1}(\zeta)|.$$

This inequality combined with (4.1) and (4.4) show that $R(\zeta - \delta)$ and $R(\zeta)$ have different signs if

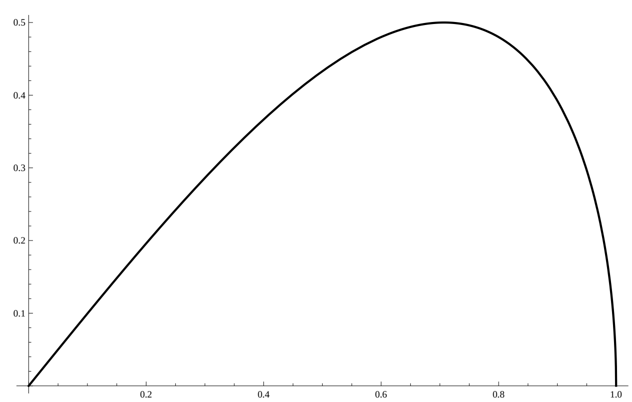
$$A > \frac{8}{5}.$$

Thus $R(x)$ has a root in the interval $(\zeta - \delta, \zeta)$.

Next we will show that δ exists with respect to all the imposed conditions and that its value is small as soon as ζ is not the smallest positive root of $U'_{2n-1}(x)$. Let ζ be such a root. From (4.2), (4.3) and (4.8), it suffices to show ζ satisfies the identity

$$\frac{A}{(2n-1)(2n+1)} < \min\left(\frac{1}{4n}\zeta\sqrt{1-\zeta^2}, \frac{\zeta^2}{6}\right). \tag{4.9}$$

The graph of the function $\zeta\sqrt{1-\zeta^2}$ is given below:



From this graph, it suffices to show that ζ is not too small and not too close to 1. Let

$\zeta_1 > 0$ be the second smallest root of $U'_{2n-1}(x)$. This root has an upper bound

$$\zeta_1 < \cos\frac{\pi(n-2)}{2n} \sim \frac{\pi}{n}$$

since the right side is the second smallest positive root of $U_{2n-1}(x)$. The smallest positive root of $U_{2n-1}(x)$ is

$$\cos\frac{\pi(n-1)}{2n} \sim \frac{\pi}{2n}.$$

Next the trivial lower bound $\zeta_1 > \pi/2n$ does not guarantee the existence of A in (4.9). We need a slightly better lower bound.

For some $\xi \in (\pi/2n, \zeta_1)$ we have

$$(\zeta_1 - \frac{\pi}{2n})U'_{2n-1}(\xi) = U_{2n-1}(\zeta_1).$$

The bound (4.6) gives

$$\begin{aligned} \zeta_1 - \frac{\pi}{2n} &> \frac{|U_{2n-1}(\zeta_1)|(1-\xi^2)}{2(2n-1)} \\ &> \frac{1-(\pi/n)^2}{2(2n-1)} \\ &\sim \frac{1}{4n}, \end{aligned}$$

so

$$\zeta_1 > \frac{2\pi+1}{4n}.$$

We now see that for n large ζ_1 satisfies the condition (4.9) if

$$A < \frac{2\pi+1}{4}.$$

Similarly, let ζ_2 be the largest root of $U'_{2n-1}(x)$. The trivial upper bound $\zeta_2 < \cos\pi/2n$, the largest root of $U_{2n-1}(x)$, does not guarantee the existence of A. However for some

$\xi \in (\zeta_2, \cos\pi/2n)$ we have

$$(\cos\frac{\pi}{2n} - \zeta_2)^2 U''_{2n-1}(\xi) = U_{2n-1}(\zeta_2).$$

The approximation (4.7) yields

$$\begin{aligned}(\cos\frac{\pi}{2n} - \zeta_2)^2 &\sim \frac{|U_{2n-1}(\zeta_2)|(1-\xi^2)}{4n^2|U_{2n-1}(\xi)|} \\ &> \frac{(1-\cos^2\pi/2n)}{4n^2} \\ &\sim \frac{\pi^2}{16n^4}.\end{aligned}$$

Thus

$$\begin{aligned}\sqrt{1-\zeta_2^2} &> \sqrt{1-(\cos\frac{\pi}{2n} - \frac{\pi}{4n^2})^2} \\ &\sim \sqrt{\frac{\pi^2}{4n^2} + \frac{\pi}{2n^2}} \\ &> \frac{2\pi+1}{4n}.\end{aligned}$$

Now the condition(4.9) is satisfied provided that $A < (2\pi+1)/4$.

Thus we can let A be any value in the interval $(8/5, (2\pi+1)/4)$. Since for large n, A and δ are arbitrarily bigger than $8/5$ and $A/4n^2\zeta$ respectively, we have that for every root ζ of $U'_{2n-1}(x)$ except the smallest root, there is a root of $R(x)$ in the small interval

$$[\zeta - \frac{8}{20n^2\zeta}, \zeta).$$

This implies that in each pair, the root ζ^2 of $U'_{2n-1}(\sqrt{x})$ is bigger than the root of x_i of $M(x)$ by at most

$$\frac{8}{20n^2\zeta}(\zeta + \sqrt{x_i}) < \frac{8}{10n^2}.$$

Combining this fact with (4.1), we conclude that for every root x_0 of $J(x)$ except the smallest root, there is a root of $M(x)$ in the interval

$$[x_0 - \frac{3}{10n^2}, x_0 + \frac{1}{2n^2})$$

for large n. This concludes the proof.

Remark: Similar Mutt and Jeff phenomena seem to occur for the integral transform of Legendre polynomials.

Chapter 5

Roots of polynomials and their generating functions: A specific example

The goal of this chapter and the next chapter is to analyze the connection between the root distribution of a sequence of polynomials, say $H_n(x)$, and the discriminant of the denominator of its generating function. In several cases, the roots of $H_n(x)$ all lie on the union of some fixed algebraic curves whose endpoints are given by the roots of the discriminant of the denominator of its genenerating function. In this chapter we will start with a specific sequence of polynomials and analyze the generating function. In Chapter 6, we will start with some general generating functions and analyze the root distribution of the generated polynomials. In the present, we focus on the sequences of polynomials given by

$$H_m^{(n)}(x) := \prod_{k=1}^{m}(x + (2\cos k\theta_m + 2)^n)$$

where

$$\theta_m = 2\pi/(2m+1).$$

As we can see from the definition of these polynomials, the roots all lie in the fixed interval $[-4^n, 0]$. Our goal is to find the generating functions of these sequences of polynomials. We will also prove that the denominators of the generating functions admit 0 and -4^n as roots.

See Table 1 for a short tabulation of these polynomials for $1 \leq n \leq 3$. It immediately suggests that for $n = 1$ we have a familiar type of Fibonacci polynomial (for $x = 1$ the values are $1, 2, 5, 13, 34$, etc.). However, for $n \geq 2$ the nature of these polynomials is much less obvious.

Before proceeding, we recall some basic facts about hypergeometric series and Wilf-

- $n = 1$

$$H_0^{(1)}(x) = 1$$
$$H_1^{(1)}(x) = 1 + x$$
$$H_2^{(1)}(x) = 1 + 3x + x^2$$
$$H_3^{(1)}(x) = 1 + 6x + 5x^2 + x^3$$
$$H_4^{(1)}(x) = 1 + 10x + 15x^2 + 7x^3 + x^4$$

- $n = 2$

$$H_0^{(2)}(q) = 1$$
$$H_1^{(2)}(q) = 1 + q$$
$$H_2^{(2)}(q) = 1 + 7q + q^2$$
$$H_3^{(2)}(q) = 1 + 26q + 13q^2 + q^3$$
$$H_4^{(1)}(q) = 1 + 70q + 87q^2 + 19q^3 + q^4$$

- $n = 3$

$$H_0^{(3)}(x) = 1$$
$$H_1^{(3)}(x) = 1 + x$$
$$H_2^{(3)}(x) = 1 + 18x + x^2$$
$$H_3^{(3)}(x) = 1 + 129x + 38x^2 + x^3$$
$$H_4^{(3)}(x) = 1 + 571x + 627x^2 + 58x^3 + x^4$$

Table 5.1: The polynomials $H_m^{(n)}(x)$

Zeibeger's algorithm. A hypergeometric series is defined as

$$ {}_pF_q\left(\begin{matrix} a_1, a_2, \ldots, a_p \\ \\ b_1, b_2, \ldots, b_q \end{matrix}; x\right) := \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \frac{x^n}{n!} $$

where $|x| < 1$. Euler obtained the following contiguous relation [1, equation (2.5.3)] for a hypergeometric series ${}_2F_1$:

$$ \begin{aligned} {}_2F_1\left(\begin{matrix} a, b \\ \\ c \end{matrix}; u\right) &= \frac{c + (a - b + 1)u}{c} {}_2F_1\left(\begin{matrix} a+1, b \\ \\ c+1 \end{matrix}; u\right) \\ &\quad - \frac{(a+1)(c-b+1)u}{c(c+1)} {}_2F_1\left(\begin{matrix} a+2, b \\ \\ c+2 \end{matrix}; u\right). \end{aligned} $$

A Wilf-Zeilberger pair (F, G) [10] satisfies the equation

$$ F(m+1, i) - F(m, i) = G(m, i+1) - G(m, i). $$

By telescoping summation, if $G(m, a) = G(m, b+1) = 0$ then $\sum_{i=a}^{b} F(m, i)$ does not depend on m. Before focusing on the generating functions of $H_m^{(n)}(x)$, we digress by showing that these polynomials arise naturally from the discriminant of the product of two rather simple polynomials

$$ K_n(x, q) = (1 + x)^{2n} + qx^n $$

and

$$ f_m(x) = (x^{2m+1} - 1)/(x - 1). $$

This was first observed by Stolarsky [14]. For a published version of the determination of the generating functions, see [3].

5.1 A general form of the discriminant

We first note that in the cases $q = 0$ and $q = -2^{2n}$ the polynomial $K_n(x, q)$ has multiple roots. These roots arise solely from the $K_n(x, q)$ polynomial. So one expects that $\Delta_x(K_n f_m)$ has factors q and $(q + 2^{2n})$. Other factors of $\Delta_x(K_n f_m)$ appear when $K_n(x, q)$ and $f_m(x)$ have common roots.

Proposition 5.1. *The discriminant of $K_n(x, q)$ is given by*

$$\mathrm{Disc}_x(K_n(x, q)) = n^{2n} q^{2n-1}(q + 2^{2n}).$$

Proof. Let x_l be a root of $K_n(x)$ (we suppress the parameter q for a moment). Then

$$\begin{aligned}
(1 + x_l)K_n'(x_l) &= 2n(1 + x_l)^{2n} + nqx_l^{n-1}(1 + x_l) \\
&= -2nqx_l^n + nqx_l^{n-1}(1 + x_l) \\
&= nqx_l^{n-1}(1 - x_l).
\end{aligned}$$

Since $(-1)^{2n(2n-1)/2} = (-1)^n$, it follows that

$$(-1)^n \mathrm{Disc}_x(K_n(x, q)) \prod_{l=1}^{2n}(1 + x_l) = n^{2n} q^{2n} \prod_{l=1}^{2n}(1 - x_l).$$

Moreover we have

$$\prod_{l=1}^{2n}(1 + x_l) = K_n(-1, q) = (-1)^n q$$

and

$$\prod_{l=1}^{2n}(1 - x_l) = K_n(1, q) = q + 4^n,$$

and the proposition follows.

$\square$

Proposition 5.2. *The discriminant of $f_m(x)$ is*

$$\mathrm{Disc}_x f_m(x) = (-1)^m (2m+1)^{2m-1}.$$

Proof. According to (1.4) we have

$$\begin{aligned}\mathrm{Disc}_x f_m(x) &= (-1)^{2m(2m-1)/2} \prod_{k=1}^{2m} f'(e^{ik\theta_m}) \\ &= (-1)^m \prod_{k=1}^{2m} \frac{2m+1}{e^{ik\theta_m}-1}.\end{aligned}$$

Since the denominator factors are the nonzero roots of $(x+1)^{2m+1} = 1$, their product is $2m+1$, and the proposition follows.

□

Proposition 5.3. *The discriminant of $K_n(x,q)f_m(x)$ is*

$$\begin{aligned}\mathrm{Disc}_x(K_n f_m) &= C_m^{(n)} q^{2n-1}(q+2^{2n}) \prod_{k=1}^{m} (q + (2\cos k\theta_m + 2)^n)^4 \\ &= C_m^{(n)} q^{2n-1}(q+2^{2n}) H_m^{(n)}(q)^4\end{aligned}$$

where

$$C_m^{(n)} = (-1)^m (2m+1)^{2m-1} n^{2n}.$$

Proof. From the definition of discriminant we note that

$$\mathrm{Disc}_x(K_n f_m) = \mathrm{Disc}_x(K_n)\mathrm{Disc}_x(f_m) \prod_{\substack{1\le k\le 2m \\ 1\le l\le 2n}} (x_l - e^{ik\theta_m})^2$$

where the x_l's are roots of K_n. Thus by Proposition 5.2 and Proposition 5.1 it suffices to show that

$$\begin{aligned}
\prod_{\substack{1\leq k\leq 2m\\ 1\leq l\leq 2n}} (x_l - e^{ik\theta_m})^2 &= \prod_{k=1}^{m} (q + (2\cos k\theta + 2)^n)^4 \\
&= \prod_{k=1}^{2m} (q + (e^{ik\theta_m} + e^{-ik\theta_m} + 2)^n)^2.
\end{aligned}$$

From the definition of K_n, we can list all roots of this polynomial in pairs (x_l, x_l^{-1}), so that

$$e^{il\phi_n + i\pi/n}\sqrt[n]{|q|} = x_l + x_l^{-1} + 2$$

where $\phi_n = 2\pi/n$ and $1 \leq l \leq n$. Thus

$$\begin{aligned}
q + (e^{ik\theta_m} + e^{-ik\theta_m} + 2)^n &= (e^{ik\theta_m} + e^{-ik\theta_m} + 2)^n - (e^{i\pi/n}\sqrt[n]{|q|})^n \\
&= \prod_{l=1}^{n} ((e^{ik\theta_m} + e^{-ik\theta_m} + 2) - e^{il\theta_n + i\pi/n}\sqrt[n]{|q|}) \\
&= \prod_{l=1}^{n} ((e^{ik\theta_m} + e^{-ik\theta_m} + 2) - (x_l + x_l^{-1} + 2)) \\
&= \prod_{l=1}^{n} \frac{(x_l - e^{ik\theta_m})(x_l^{-1} - e^{ik\theta_m})}{e^{ik\theta_m}}.
\end{aligned}$$

Since $\prod_{i\leq k\leq 2m} e^{ik\theta_m} = 1$, the proposition follows.

□

Corollary 5.4. *For any m, the $H_{3m+1}^{(n)}(q)$ polynomials are divisible by $q+1$.*

Proof. Consider

$$H_{3m+1}^{(n)}(q) = \prod_{k=1}^{3m+1} (q + (2\cos k\theta_{3m+1} + 2)^n)$$

where

$$\theta_{3m+1} = \frac{2\pi}{6m+3}.$$

When $k = 2m+1$ we have $(2\cos k\theta_{3m+1} + 2)^n = 1$. Thus $(q+1)|H_{3m+1}^{(n)}(q)$.

$\square$

Corollary 5.5. *For any* n*,*

$$H_m^{(n)}(q) | H_{3m+1}^{(n)}(q).$$

Proof. This is clear since the terms $k = 3, 6, 9, \ldots, 3m$ in the product for $H_{3m+1}^{(n)}(q)$ give $H_m^{(n)}(q)$.

$\square$

Next we will focus on the generating function of $H_m^{(n)}(x)$.

5.2 Generating function for $H_m^{(1)}(q)$

It is not hard to show that $H_m^{(1)}(q)$ has a generating function similar to that of the closely related Chebyshev polynomials. We give the details both for the sake of completeness, and because they are useful in the rather harder analysis required for $H_m^{(2)}(q)$.

Proposition 5.6. *The polynomials* $H_m^{(1)}(x)$ *satisfy*

$$\frac{1-t}{(1-t)^2 - xt} = 1 + \sum_{m=1}^{\infty} H_m^{(1)}(x) t^m.$$

Proof. Recall the generating function definition of Chebyshev T-polynomial $T_n(x)$ (see [11]):

$$\frac{1-xt}{1+t^2-2tx} = 1 + \sum_{m=1}^{\infty} T_m(x) t^m.$$

By replacing x by $(x+2)/2$, multiplying both sides by 2 and subtracting 1 from each side we obtain

$$\frac{1-t}{(1-t)^2 - xt} = \frac{1}{1+t} \left(1 + \sum_{m=1}^{\infty} 2T_m \left(\frac{x+2}{2} \right) t^m \right).$$

So it remains to prove that $H_m^{(1)}(x)$ is twice an alternating sum of Chebyshev polynomials

$T_m((x+2)/2)$ plus or minus 1 depending on the parity of m, i.e.

$$H_m^{(1)}(x) = 2T_m\left(\frac{x+2}{2}\right) - 2T_{m-1}\left(\frac{x+2}{2}\right) + 2T_{m-2}\left(\frac{x+2}{2}\right) - \cdots \mp 2T_1\left(\frac{x+2}{2}\right) \pm 1.$$

By writing $2\cos m\theta = z^m + z^{-m}$ where $z = e^{i\theta}$, we obtain

$$S_m(z) := 2\cos m\theta - 2\cos(m-1)\theta + 2\cos(m-2)\theta - \cdots \mp 2\cos\theta \pm 1 = \frac{z^{2m+1}+1}{z^m(z+1)}. \quad (5.1)$$

$S_m(z)$ has roots $e^{-ik\theta_m}$, so the twice alternating sum of Chebyshev polynomials plus or minus 1 has a root x such that

$$\frac{x+2}{2} = -\cos k\theta_m$$

or $x = -2\cos k\theta_m - 2$. These roots correspond to roots of $H_m^{(1)}(x)$. This completes the proof since both sides of (5.1) are monic polynomials in x.

□

Proposition 5.6 leads to an explicit formula for the coefficients of $H_m^{(1)}(x)$. This formula is not new, but will be useful in finding the generating function for $H_m^{(2)}(x)$.

Proposition 5.7. *The polynomial $H_m^{(1)}(x)$ is given by the formula*

$$H_m^{(1)}(x) = \sum_{k=0}^{m} \binom{m+k}{m-k} x^k.$$

Proof. Let $a_{k,m}$be the coefficients of the polynomial $H_m^{(1)}(x)$. By interchanging summation, we have the following identity

$$\begin{aligned} \sum_{m=0}^{\infty} H_m^{(1)}(x)t^m &= \sum_{m=0}^{\infty}\sum_{k=0}^{m} a_{k,m}x^k t^m \\ &= \sum_{k=0}^{\infty} x^k t^k \sum_{m=0}^{\infty} a_{k,m+k}t^m. \end{aligned}$$

On the other hand, by expanding the generating function of $H_m^{(1)}(x)$ in terms of x, we obtain

$$\begin{aligned}\frac{1-t}{(1-t)^2-xt} &= \frac{1}{1-t}\frac{1}{1-xt/(1-t)^2} \\ &= \sum_{k=0}^{\infty} x^k t^k/(1-t)^{2k+1} \\ &= \sum_{k=0}^{\infty} x^k t^k \sum_{m=0}^{\infty} \binom{m+2k}{m} t^m.\end{aligned}$$

By equating the two double summations, we find that

$$a_{k,m+k} = \binom{m+2k}{m},$$

and the proposition follows.

$\square$

5.3 Generating function for $H_m^{(2)}(x)$

Stolarsky [14] conjectured that the generating function of $H_m^{(2)}(x)$ was

$$\frac{(1-t)^3}{(1-t)^4-xt(t+1)^2}.$$

Our approach to prove this formula is to express $H_m^{(2)}(x)$ in terms of $H_m^{(1)}(x)$ and apply Proposition 5.7. In particular, the connection between $H_m^{(2)}(x)$ and $H_m^{(1)}(x)$ is given by

$$H_m^{(2)}(-x^2) = H_m^{(1)}(x)H_m^{(1)}(-x).$$

This equation easily follows from the definition of $H_m^{(n)}(x)$. We now assume the following identity

$$\sum_{i=0}^{2k}(-1)^i\binom{m+k+i}{m+k-i}\binom{m+3k-i}{m-k+i}=(-1)^k\sum_{i=0}^{m}\binom{2k}{m-i}\binom{4k+i}{i}. \tag{5.1}$$

We will provide a proof for this identity later.

Proposition 5.8. *The polynomials $H_m^{(2)}(x)$ satisfy*

$$\frac{(1-t)^3}{(1-t)^4-xt(t+1)^2}=1+\sum_{m=1}^{\infty}H_m^{(2)}(x)t^m.$$

Proof. Recall that $H_m^{(2)}(-x^2)=H_m^{(1)}(x)H_m^{(1)}(-x)$. So $H_m^{(2)}(-x^2)$ is an even polynomial of degree $2m$, whose k^{th}-coefficient is given by

$$\sum_{i=0}^{k}(-1)^i\binom{m+i}{m-i}\binom{m+k-i}{m-k+i}.$$

By interchanging summations as in the proof of Proposition 5.7, we obtain

$$\sum_{m=0}^{\infty}H_m^{(2)}(-x^2)t^m=\sum_{k=0}^{\infty}x^{2k}t^k\sum_{m=0}^{\infty}\sum_{i=0}^{2k}(-1)^i\binom{m+k+i}{m+k-i}\binom{m+3k-i}{m-k+i}t^m.$$

Upon expanding the function

$$\frac{(1-t)^3}{(1-t)^4+x^2t(t+1)^2}$$

in terms of x first and then in terms of t one obtains

$$\begin{aligned}\frac{(1-t)^3}{(1-t)^4+x^2t(t+1)^2} &= \sum_{k=0}^{\infty}(-1)^kx^{2k}t^k(t+1)^{2k}/(1-t)^{4k+1}\\ &= \sum_{k=0}^{\infty}(-1)^kx^{2k}t^k\sum_{i=0}^{2k}\binom{2k}{i}t^i\sum_{j=0}^{\infty}\binom{4k+j}{j}t^j\\ &= \sum_{k=0}^{\infty}(-1)^kx^{2k}t^k\sum_{m=0}^{\infty}\sum_{i=0}^{m}\binom{2k}{m-i}\binom{4k+i}{i}t^m.\end{aligned}$$

The proposition follows from the identity (3.1) above.

$\square$

It is interesting to note that since $H_m^{(2)}(-x^2)$ is an even function, the following identity holds for any odd integer k:

$$\sum_{i=0}^{2m}(-1)^i\binom{m+i}{m-i}\binom{m+k-i}{m-k+i}=0.$$

5.4 A hypergeometric identity from Euler's contiguous relation and the Wilf-Zeilberger algorithm

In this section we will prove identity (5.1). The method of proving this identity is similar to that of Vidunas [19]. We first express the right-hand side of this equation in terms of hypergeometric ${}_2F_1$ functions.

Proposition 5.9. *The following equations hold:*

$$\sum_{i=0}^{m}\binom{2k}{m-i}\binom{4k+i}{i}=\binom{2k}{m}{}_2F_1\left(\begin{matrix}4k+1,-m\\ 2k-m+1\end{matrix};-1\right)$$

if $2k\geq m$ *and*

$$\sum_{i=0}^{m}\binom{2k}{m-i}\binom{4k+i}{i}=\binom{m+2k}{m-2k}{}_2F_1\left(\begin{matrix}-2k,2k+m+1\\ m-2k+1\end{matrix};-1\right)$$

if $2k\leq m$.

Proof. The first equation follows from the fact that

$$\binom{4k+i}{i}=\frac{(4k+1)_i}{i!}$$

and

$$\binom{2k}{m-i} = \frac{(-1)^i(2k)!(-m)_i}{m!(2k-m)!(2k-m+1)!}.$$

For the second equation, we note that the summand on the left side equals 0 when $i < m-2k$. Thus the left side equals

$$\sum_{i=m-2k}^{m} \binom{2k}{m-i}\binom{4k+i}{4k} = \sum_{i=0}^{2k} \binom{2k}{i}\binom{4k+m-i}{4k} = \sum_{i=0}^{2k} \binom{2k}{i}\binom{m+2k+i}{4k}.$$

To complete the proof, we note that

$$\binom{2k}{i} = \frac{(-2k)_i(-1)^i}{i!}$$

and

$$\binom{2k+m+i}{4k} = \frac{(2k+m)!(2k+m+1)_i}{(4k)!(m-2k)!(m-2k+1)_i}.$$

Thus the proposition follows.

$\square$

To continue the proof of (5.1), we recall Euler's contiguous relation:

$$\begin{aligned} {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; u\right) &= \frac{c+(a-b+1)u}{c}\,{}_2F_1\left(\begin{matrix} a+1, b \\ c+1 \end{matrix}; u\right) \\ &\quad - \frac{(a+1)(c-b+1)u}{c(c+1)}\,{}_2F_1\left(\begin{matrix} a+2, b \\ c+2 \end{matrix}; u\right). \end{aligned}$$

Applying this relation to

$${}_2F_1\left(\begin{matrix} 4k+1, -m \\ 2k-m+1 \end{matrix}; -1\right)$$

with $a=-m$, $b=4k+1$ and $c=2k-m+1$, we obtain the identity

$$\begin{aligned}{}_2F_1\left(\begin{matrix}4k+1,-m\\ \\ 2k-m+1\end{matrix};-1\right) &= \frac{6k+1}{2k-m+1}{}_2F_1\left(\begin{matrix}4k+1,-m+1\\ \\ 2k-m+2\end{matrix};-1\right)\\ &\quad+\frac{(m-1)(m-1+2k)}{(2k-m+1)(2k-m+2)}{}_2F_1\left(\begin{matrix}4k+1,-m+2\\ \\ 2k-m+3\end{matrix};-1\right).\end{aligned}$$

Also applying the same contiguous relation to

$${}_2F_1\left(\begin{matrix}-2k,2k+m+1\\ \\ m-2k+1\end{matrix};-1\right)$$

with $a=2k+m+1$, $b=-2k$ and $c=m-2k+1$, we obtain

$$\begin{aligned}{}_2F_1\left(\begin{matrix}-2k,2k+m+1\\ \\ m-2k+1\end{matrix};-1\right) &= -\frac{6k+1}{m-2k+1}{}_2F_1\left(\begin{matrix}-2k,2k+m+2\\ \\ m-2k+2\end{matrix};-1\right)\\ &\quad+\frac{(m+2)(2k+m+2)}{(m-2k+1)(m-2k+2)}{}_2F_1\left(\begin{matrix}-2k,2k+m+3\\ \\ m-2k+3\end{matrix};-1\right).\end{aligned}$$

Fix k and denote by S_m the right side of (5.1). From the two identities above and Proposition 5.9, we have the recursive relation

$$(m+2)S_{m+2}=(6k+1)S_{m+1}+(m+1+2k)S_m,$$

where $S_0=(-1)^k$ and $S_1=(-1)^k(6k+1)$.

Let T_m be the left side of (5.1). It is easy to check that $T_0=(-1)^k$ and

$$\begin{aligned}T_1 &= (-1)^{k-1}\binom{2k}{2}+(-1)^k(2k+1)^2+(-1)^{k+1}\binom{2k}{2}\\ &= (-1)^k(6k+1).\end{aligned}$$

So it suffices to show that the sequence T_m also satisfies the same recursive relation, i.e.

$$(m+2)T_{m+2} - (6k+1)T_{m+1} - (m+1+2k)T_m = 0. \tag{5.1}$$

By the definition of T_m, the left hand side of the relation above is a finite summation of several terms. We denote by $f(m,i)$ its summand. Also let

$$F(m,i) = f(m,i)/(2k+m+1)!.$$

We apply the WZ-algorithm with the certificate function

$$R(m,i) := \frac{i(2i-1)(i-3k-m-1)}{2(-3+i-k-m)(2+2k+m)} \frac{R_1(m,i)}{R_2(m,i)}$$

where

$$\begin{aligned} R_1(m,i) &= -10 - i + i^2 - 5k - 4ik + 2i^2k + 7k^2 - 4ik^2 \\ &\quad +2k^3 - 13m - im + i^2m - 5km \\ &\quad -2ikm + k^2m - 6m^2 - 2km^2 - m^3 \end{aligned}$$

and

$$\begin{aligned} R_2(m,i) &= -5i^2 + 2i^4 + 2k + 10ik - 3i^2k - 8i^3k \\ &\quad -5k^2 + 6ik^2 + 6i^2k^2 - 11k^3 + 4ik^3 - 4k^4 \\ &\quad -6i^2m + 3km + 12ikm - 4i^2km - 10k^2m + 8ik^2m \\ &\quad -8k^3m - 2i^2m^2 + km^2 + 4ikm^2 - 4k^2m^2. \end{aligned}$$

With computer algebra, one can verify that

$$F(m+1,i) - F(m,i) = G(m,i+1) - G(m,i)$$

where $G(m,i) = R(m,i)F(m,i)$. The equation above can be checked easily by Mathematica using the FactorialSimplify function in the aisb.m package (see [9]). We note that $G(m,0) = 0$ since $R(m,0) = 0$ by definition of $R(m,i)$. Also $G(m,2k+1) = 0$ since $F(m,2k+1) = 0$. Therefore

$$\sum_{i=0}^{2k} F(m,i)$$

is a constant and it is easy to check that this constant is 0 by the initial condition. Thus

$$\sum_{i=0}^{2k} f(m,i) = 0,$$

and T_m satisfies the recursive relation (5.1).

5.5 Generating function for $H_m^{(n)}(x)$

In this section we will provide an algorithm to produce the generating functions for $H_m^{(n)}(x)$, which was defined previously as

$$H_m^{(n)}(q) := \prod_{k=1}^{m} (q + (2\cos k\theta_m + 2)^n)$$

where

$$\theta_m = 2\pi/(2m+1).$$

For convenience, we work with the polynomial $G_m^{(n)}(x) = (-1)^m H_m^{(n)}(-x)$. It is monic, with roots equal to the n-th powers of the roots of $G_m^{(1)}(x) = (-1)^m H_m^{(1)}(-x)$. Let ϵ_n be a primitive

n-th root of unity. Then

$$G_m^{(n)}(x^n) = \prod_{G_m^{(1)}(\alpha)=0} (x^n - \alpha^n) = \prod_{G_m^{(1)}(\alpha)=0} \left(\prod_{j=1}^{n-1} (x - \alpha/\epsilon_n^j) \right),$$

which equals

$$\begin{aligned} \prod_{j=0}^{n-1} \left(\prod_{G_m^{(1)}(\alpha)=0} (x - \alpha/\epsilon_n^j) \right) &= \prod_{j=0}^{n-1} \epsilon_n^{-jm} \prod_{j=0}^{n-1} G_m^{(1)}(\epsilon_n^j x) \\ &= (-1)^{m(n-1)} \prod_{j=0}^{n-1} G_m^{(1)}(\epsilon_n^j x). \end{aligned}$$

Therefore, the polynomial $G_m^{(n)}$ can be expressed using only $G_m^{(1)}$ by

$$G_m^{(n)}(x^n) = (-1)^{m(n-1)} \prod_{j=0}^{n-1} G_m^{(1)}(\epsilon_n^j x), \tag{5.1}$$

and in particular it is given by the resultant $G_m^{(n)}(x) = \mathrm{Res}_y(G_m^{(1)}(y), x - y^n)$.

A consequence of Proposition 5.6 is that the sequence of polynomials $G_m^{(1)}(x)$ satisfies the linear recurrence

$$G_{m+2}^{(1)}(x) + (2 - x) G_{m+1}^{(1)}(x) + G_m^{(1)}(x) = 0 \tag{5.2}$$

with $G_0^{(1)}(x) = 1$ and $G_1^{(1)}(x) = x - 1$. The coefficients of this recurrence are "constant", in the sense that they do not depend on the index m. The characteristic polynomial of the sequence $G_m^{(1)}(x)$ is $t^2 + (2 - x)t + 1$ which is the denominator of its generating function.

Conversely, any sequence satisfying a linear recurrence with constant coefficients admits a rational generating function and this applies in particular to the sequence of polynomials $G_m^{(1)}(\epsilon_n^j x)$ for any $j \geq 0$.

The product of sequences u_n and v_n that are solutions of linear recurrences with constant coefficients satisfies a linear recurrence with constant coefficients (see, e.g. [18, Sec. 2.4], [13, Prop. 4.2.5], [4, Sec. 2, Ex.5]). Its generating function, called the Hadamard product of those of $u_n$and v_n, is therefore rational. From (5.1), it follows that the generating function $\sum_{m\geq 0} G_m^{(n)}(x^n)t^m$ is also rational.

Moreover, the reciprocal of the denominator of a Hadamard product of rational series has roots as the pairwise products of those of the individual series. Thus by letting $\alpha_1(x)$, $\alpha_2(x)$ be the roots of $(1+t)^2 - xt$, the reciprocal of the denominator of the generating function of $(-1)^{m(n-1)}G_m^{(n)}(x^n)$ is the characteristic polynomial

$$P_n(x,t) = \prod_{1\leq i_1,\dots,i_n\leq 2} \left(t - \alpha_{i_1}(x)\alpha_{i_2}(\epsilon_n x)\cdots\alpha_{i_n}(\epsilon_n^{n-1}x)\right).$$

Note that, since $\alpha_1\alpha_2 = 1$, the polynomial P_n is self-reciprocal with respect to t.

We prove that the polynomial $P_n(x,t)$ belongs to $\mathbb{Q}[x^n,t]$ by showing that all the (Newton) power sums of the roots of $P_n(x,t)$ belong to $\mathbb{Q}[x^s]$. For any $l \in \mathbb{N}$, the l-th power sum is equal to the product

$$(\alpha_1^l(x) + \alpha_2^l(x))(\alpha_1^l(\epsilon_n x) + \alpha_2^l(\epsilon_n x))\cdots(\alpha_1^l(\epsilon_n^{n-1}x) + \alpha_2^l(\epsilon_n^{n-1}x)) \tag{5.3}$$

and thus has the form $T_{l,n}(x) = Q_l(x)Q_l(\epsilon_n x)\cdots Q_l(\epsilon_n^{n-1}x)$ for some polynomial $Q_l(x) \in \mathbb{Q}[x]$. The polynomial $T_{l,n}(x)$ is symmetric in $x, \epsilon_n x, \dots, \epsilon_n^{n-1}x$ and thus it belongs to $\mathbb{Q}[x^n]$, and so do all the coefficients of P_n. This can also be seen from the expression of $T_{l,s}$ as a resultant: $T_{l,s}(x) = \mathrm{Res}_y(y^n - x^n, Q_l(y))$.

In conclusion, $G_m^{(n)}(x^n)$ satisfies a recurrence with coefficients that are polynomials in $\mathbb{Q}[x^n]$, thus the series $\sum_{m\geq 0} G_m^{(n)}(x^n)t^m$ is rational and belongs to $\mathbb{Q}(x^n,t)$, and thus the series $\sum_{m\geq 0} H_m^{(n)}(x)t^m$ belongs to $\mathbb{Q}(x,t)$.

We will first illustrate the ideas of the algorithm to produce the generating function in the simplest case $n=2$, for which the computations can be done by hand. We compute the generating function of $k_m(x) := H_m^{(2)}(-x^2) = G_m^{(1)}(x)G_m^{(1)}(-x)$ from which that of the sequence $H_m^{(2)}(x)$ is easily deduced. By Proposition 5.6, the generating functions of the sequences $G_m^{(1)}(x)$ and $G_m^{(1)}(-x)$ have denominators

$$\begin{aligned} g_1(t) &= (1+t)^2 - xt \\ g_2(t) &= (1+t)^2 + xt, \end{aligned}$$

which are self-reciprocal. The polynomial whose roots are the product of the roots of g_1 and g_2 is obtained by a resultant computation:

$$\mathrm{Res}_u(g_1(u), u^2 g_2(t/u)) = (t-1)^4 + x^2 t(1+t)^2 = 1 + (x^2-4)t + (2x^2+6)t^2 + (x^2-4)t^3 + t^4.$$

It follows that the sequence $k_m(x)$ satisfies the fourth order recurrence

$$k_{m+4} + (x^2-4)k_{m+3} + (2x^2+6)k_{m+2} + (x^2-4)k_{m+1} + k_m = 0.$$

The initial conditions can be determined separately

$$k_0 = 1,\ k_1 = 1 - x^2,\ k_2 = 1 - 7x^2 + x^4,\ k_3 = 1 - 26x^2 + 13x^4 - x^6$$

and these can be used to compute the numerator of the generating function of the sequence of polynomial $H_m^{(2)}(x)$. In conclusion, we have just proved the identity

$$\sum_{m=0}^{\infty} H_m^{(2)}(x) t^m = \frac{(1-t)^3}{(1-t)^4 - xt(t+1)^2}.$$

In general, we can obtain the generating function of $H_m^{(n)}(x)$ using the following algorithm:

Input: an integer $n \geq 2$.

Output: the generating function of $H_m^{(n)}(x)$.

(1) Compute the powersums $Q_l(x) = \alpha_1(x)^l + \alpha_2(x)^l \in \mathbb{Q}[x]$ for $l = 0, \ldots, 2^n$

(2) Compute the powersums $T_{l,n}$ from (5.3) by $T_{l,n}(x^{1/n}) = \mathrm{Res}_y(y^s - x, Q_l(y))$ for $l =$

$0, \ldots, 2^n$

(3) Recover the polynomial $P_n(x^{1/n}, t)$ from its powersums $T_{l,n}(x^{1/n}, t)$

(4) Deduce the denominator $D_n(x,t)$ using $P_n(x^{1/n}, t) = D_n(-x, (-1)^n t)$

(5) Compute $G_m^{(1)}(x)$ for $m = 0, \ldots, 2^n - 1$ using the 2nd order recurrence (5.2)

(6) Compute $G_m^{(n)}(x) = \mathrm{Res}_y(G_m^{(1)}(y), x - y^n)$ for $m = 0, \ldots, 2^n - 1$

(7) Compute the numerator $N_n(x,t) = D_n(x,t) \times \sum_{m=0}^{2^n-1} G_m^{(n)}(-x)(-t)^m \bmod t^{2^n}$

(8) Return the rational function $N_n(x,t)/D_n(x,t)$

For $n = 3$, we obtain the following result

$$\sum_{m\geq 0} H_m^{(3)}(x)t^m = \frac{(1-t)((t-1)^6 - xt^2(t+3)(3t+1))}{x^2t^4 - xt(t^4+14t^3+34t^2+14t+1)(t-1)^2 + (t-1)^8},$$

which was conjectured by Stolarsky in [14]. For $n = 4$, the generating function $\sum_{m\geq 0} H_m^{(4)}(x)t^m$ is

$$\frac{(t-1)(x^2t^4A(t) - 2xt^2B(t)(t-1)^6 + (t-1)^{14})}{x^3t^5(t+1)^2(t-1)^4 + x^2t^3C(t) + xt(t-1)^8D(t) - (t-1)^{16}},$$

where

$$\begin{aligned}
A(t) &= 9t^6 - 46t^5 - 89t^4 - 260t^3 - 89t^2 - 46t + 9 \\
B(t) &= 11t^4 + 128t^3 + 266t^2 + 128t + 11 \\
C(t) &= 2t^{10} - 13t^9 + 226t^8 - 300t^7 - 676t^6 - 2574t^5 - 676t^4 - 300t^3 + 226t^2 - 13t + 2 \\
D(t) &= t^6 + 60t^5 + 519t^4 + 1016t^3 + 519t^2 + 60t + 1.
\end{aligned}$$

As n increases, the generating function become more complicated. However we can show that for any n, the endpoints of the interval containing the roots of $H_m^{(n)}(x)$ which are $0, -4^n$ are the roots of the discriminant in t of the denominator of the generating function.

To illustrate this phenomenon, we consider the case $n = 3$. The discriminant in t of the denominator of the generating function is

$$\begin{aligned} & \mathrm{Disc}_t(x^2t^4 - xt(t^4 + 14t^3 + 34t^2 + 14t + 1)(t-1)^2 + (t-1)^8) \\ = \ & 729x^{10}(x + 4^2)^2(x - 4^3)^4(x + 4^3)^4. \end{aligned}$$

We summarize this phenomenon in the following theorem.

Theorem 5.10. *Let $H_m^{(n)}(x)$ be the sequence of polynomials given by*

$$H_m^{(n)}(x) := \prod_{k=1}^{m}(x + (2\cos k\theta_m + 2)^n),$$

where

$$\theta_m = 2\pi/(2m+1).$$

Then the discriminant in t of the denominator of the generating function $\sum H_m^{(n)}(x)t^m$ admits 0 and -4^n as roots.

To prove this, we look at another way of finding the generating function for $H_m^{(n)}(x)$ using the index operator. We illustrate the idea in the case $n = 3$, and the general case follows in a similar manner. As we have observed above,

$$H_m^{(3)}(x^3) = H_m^{(1)}(x)H_m^{(1)}(\omega x)H_m^{(1)}(\omega^2 x)$$

where ω is a third root of unity. We define the shifting operators M, N, K acting on $H_m^{(3)}(x^3)$ as

$$\begin{aligned} M(H_m^{(1)}(x)H_m^{(1)}(\omega x)H_m^{(1)}(\omega^2 x)) &= H_{m-1}^{(1)}(x)H_m^{(1)}(\omega x)H_m^{(1)}(\omega^2 x), \\ N(H_m^{(1)}(x)H_m^{(1)}(\omega x)H_m^{(1)}(\omega^2 x)) &= H_m^{(1)}(x)H_{m-1}^{(1)}(\omega x)H_m^{(1)}(\omega^2 x), \\ K(H_m^{(1)}(x)H_m^{(1)}(\omega x)H_m^{(1)}(\omega^2 x)) &= H_m^{(1)}(x)H_m^{(1)}(\omega x)H_{m-1}^{(1)}(\omega^2 x). \end{aligned}$$

It is clear that these operators commute and

$$MNK(H_m^{(3)}(x^3)) = H_{m-1}^{(3)}(x^3).$$

From the generating function of $H_m^{(1)}(x)$, the three operators M, N, K satisfy the following equations

$$\begin{aligned} M^2 - (2+x)M + 1 &= 0 \\ N^2 - (2+\omega x)N + 1 &= 0 \\ K^2 - (2+\omega^2 x)K + 1 &= 0. \end{aligned}$$

Our goal is to derive the minimal polynomial of MNK with coefficients in $\mathbb{Q}[x]$. From the definition of discriminant, we notice that in general if $t_1$and t_2 are roots of some polynomials $P_1(t)$ and $P_2(t)$ respectively, then t_1t_2 is a root of

$$\mathrm{Disc}_x(x^{\deg P_1} P_1(t/x)P_2(x)).$$

Furthermore if either $P_1(t)$ or $P_2(t)$ has a double root, then so does $\mathrm{Disc}_x(x^{\deg P_1} P_1(t/x)P_2(x))$. Since, M, N and K are the roots of the polynomials given above, we apply the discriminant process twice to obtain the minimal polynomial of MNK. The denominator of the generating function can be obtained by replacing x^3 by x from the minimal polynomial of MNK. If $x = -4$ then the minimal polynomial of M has a double root, and so does the minimal polynomial of MNK. Thus if $x = -4^3$, the denominator of the generating function will have a double root. The result follows.

Chapter 6

Roots of polynomials and their generating functions: A general approach

In this chapter, we study the root distribution of a sequence of polynomials $H_m(z)$ whose generating function $\sum_{m=0}^{\infty} H_m(z)t^m$ is the reciprocal of a bivariate polynomial $D(z,t)$. We will show that in the three cases $D(z,t) = A(z)t^2 + B(z)t + C(z)$, $D(z,t) = t^3 + A(z)t + B(z)$ and $D(z,t) = t^4 + A(z)t + B(z)$, where $A(z), B(z)$ and $C(z)$ are any polynomials in z, the roots of $H_n(z)$ lie on some fixed algebraic curves whose equations are explicitly given.

Theorem 6.1. *Let*

$$\frac{1}{A(z)t^2 + B(z)t + C(z)} = \sum H_m(z)t^m.$$

The roots of $H_m(z)$ which satisfy $A(z)C(z) \neq 0$ and $B^2(z) - 4A(z)C(z) \neq 0$ lie and are dense as $m \to \infty$ on the curve $\mathcal{C}_2$ defined by

$$\Im \frac{\mathrm{Disc}_t(A(z)t^2 + B(z)t + C(z))}{A(z)C(z)} = 0$$

and limited by

$$-4 \leq \Re \frac{\mathrm{Disc}_t(A(z)t^2 + B(z)t + C(z))}{A(z)C(z)} \leq 0.$$

Proof. Suppose z is a root of $H_m(z)$ which satisfies $A(z)C(z) \neq 0$ and $B^2(z) - 4A(z)C(z) \neq 0$. Let t_1 and t_2 be the roots of $A(z)t^2 + B(z)t + C(z)$. By partial fractions, we have

$$\begin{aligned}
\frac{1}{A(z)t^2 + B(z)t + C(z)} &= \frac{1}{A(z)(t - t_1)(t - t_2)} \\
&= \frac{1}{A(z)(t_1 - t_2)} \left(\frac{1}{t - t_1} - \frac{1}{t - t_2} \right)
\end{aligned}$$

$$= \quad \frac{1}{A(z)} \sum_{m=0}^{\infty} \frac{t_1^{m+1} - t_2^{m+1}}{(t_1 - t_2) t_1^{m+1} t_2^{m+1}} t^n. \tag{6.1}$$

Thus if we let $t_1 = qt_2$ then q is an $(m+1)$-st root of unity and $q \neq 1$. By the definition of q-discriminant in (1.2), q is a root of $\mathrm{Disc}_t(A(z)t^2 + B(z)t + C(z); q)$ which equals

$$q \left(B^2(z) - (q + q^{-1} + 2) A(z) C(z) \right).$$

This implies that

$$\frac{B^2(z)}{A(z)C(z)} = q + q^{-1} + 2.$$

Thus $z \in \mathcal{C}_2$ since q is an $(m+1)$-th root of unity.

To show the density of the roots of $H_m(z)$, we let $\zeta \in \mathcal{C}$ and U be an open neighborhood of ζ such that the rational function $B^2(z)/A(z)C(z)$ is analytic on U. Since the set of $(m+1)$-th roots of unity is dense on the unit circle as $m \to \infty$, the map $B^2(z)/A(z)C(z)$ maps U to an open set containing some point $2\Re q + 2$ where q is an $(m+1)$-th root of unity for some large m. Thus there is some point $z \in U$ such that

$$\frac{B^2(z)}{A(z)C(z)} = q + q^{-1} + 2$$

or

$$\mathrm{Disc}_t(A(z)t^2 + B(z)t + C(z); q) = 0.$$

This implies that q is the quotient of the two roots of $A(z)t^2 + B(z)t + C(z)$. Hence z is a root of $H_m(z)$ by (6.1).

□

Example. *We consider an example in which the generating function of $H_n(z)$ is given by*

$$\frac{1}{z^2t^2 + (z^2 - 2z + a)t + 1} = \sum_{m=0}^{\infty} H_m(z) t^m$$

where $a \in \mathbb{R}$ *. Let* $z = x + iy$*. We exhibit the three possible cases for the root distribution of* $H_m(z)$ *depending on* a*:*

(1) *If* $a \leq 0$*, the roots of* $H_m(z)$ *lie on the two real intervals defined by*

$$(x^2 + a)(x^2 - 4x + a) \leq 0.$$

(2) *If* $0 < a \leq 4$*, the roots of* $H_m(z)$ *can lie either on the half circle* $x^2 + y^2 = a$*,* $x \geq 0$*, or on the real interval defined by* $x^2 - 4x + a \leq 0$*.*

(3) *If* $a > 4$*, the roots of* $H_m(z)$ *lie on two parts of the circle* $x^2 + y^2 = a$ *restricted by* $0 \leq x \leq 2$*.*

By complex expansion, we have

$$\begin{aligned}
\Im \frac{B^2(z)}{A(z)C(z)} &= \frac{2y(x^2 + y^2 - a)P}{(x^2 + y^2)^2}, \\
\Re \frac{B^2(z)}{A(z)C(z)} &= \frac{P^2 - Q^2}{(x^2 + y^2)^2},
\end{aligned}$$

where

$$\begin{aligned}
P &= ax - 2x^2 + x^3 - 2y^2 + xy^2, \\
Q &= y(x^2 + y^2 - a).
\end{aligned}$$

Theorem 6.1 yields three cases: $y = 0$*,* $x^2 + y^2 - a = 0$ *or* $P = 0$*. Since* $\Re B^2/AC \geq 0$*, all these cases give* $Q = 0$*. We note that if* $x^2 + y^2 - a = 0$ *then the condition* $\Re B^2/AC \leq 4$ *reduces to*

$$x(a + x^2 + y^2)(ax - 4x^2 + x^3 - 4y^2 + xy^2) = 4a^2 x(x - 2) \leq 0. \tag{6.2}$$

Suppose $a \leq 0$*. Then the condition* $Q = 0$ *implies that the roots of* $H_m(z)$ *are real. The* *condition* $\Re B^2/AC \leq 4$ *becomes*

$$(x^3 - 2x^2 + ax)^2 - 4x^4 = x^2(x^2 + a)(x^2 - 4x + a) \leq 0. \tag{6.3}$$

Suppose $0 < a \leq 4$. *The roots of* $H_m(z)$ *lie either on the half circle* $x^2 + y^2 - a = 0$, $x \geq 0$ *(from the inequality (6.2)), or on the real interval given by* $x^2 - 4x + a \leq 0$ *(from the inequality (6.3)). If* $a > 4$ *then the roots of* $H_m(z)$ *lie on the two parts of the circle* $x^2 + y^2 - a = 0$ *restricted by* $0 \leq x \leq 2$ *(from the inequality (6.2)).*

We notice that in this example, the inequality $\Re B^2/AC \leq 4$ *gives the endpoints of the curves where the roots of* $H_m(z)$ *lie. Thus, these endpoints are roots of* $\mathrm{Disc}_t(At^2 + Bt + C)$. *Moreover the critical values of a, which are 0 and 4, are roots of the double discriminant of the denominator*

$$\mathrm{Disc}_z \mathrm{Disc}_t(z^2t^2 + (z^2 - 2z + a)t + 1) = 4096a^3(a - 4).$$

This comes from the fact that the endpoints of the fixed curves containing the roots of $H_m(z)$ *are the roots of* $\mathrm{Disc}_t(z^2t^2 + (z^2 - 2z + a)t + 1)$. *When this discriminant has a double root as a polynomial in z, some two endpoints of the fixed curves coincide. That explains the change in the shape of the root distribution.*

Example. *We note that the coefficients of* $A(z)$, $B(z)$ *and* $C(z)$ *can be complex. For example, consider the sequence* $H_m(z)$ *which satisfies*

$$\frac{1}{z^2t^2 + (z^2 - 2z + i)t + 1} = \sum_{m=0}^{\infty} H_m(z)t^m.$$

By complex expansion, we have

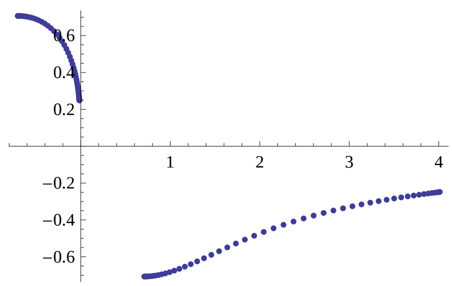
$$\Im \frac{B^2(z)}{A(z)C(z)} = \frac{2PQ}{(x^2 + y^2)^2},$$

$$\Re \frac{B^2(z)}{A(z)C(z)} = \frac{P^2 - Q^2}{(x^2+y^2)^2},$$

where

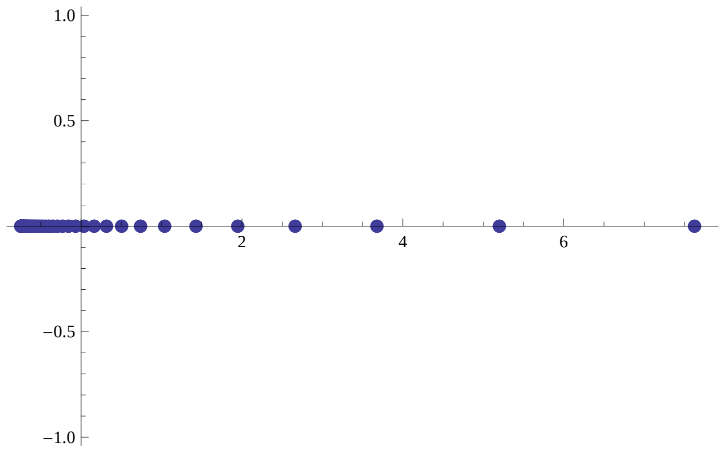
$$P = x^3 - 2x^2 + xy^2 - 2y^2 + y,$$
$$Q = y^3 + x^2y + x.$$

The conditions $\Im B^2/AC = 0$ *and* $\Re B^2/AC \geq 0$ *imply that* $y^3 + x^2y + x = 0$*. Thus the roots of* $H_m(z)$ *lie on parts of the curve* $y^3 + x^2y + x = 0$ *restricted by* $\Re B^2/AC = 0$*. The roots of* $H_{50}(z)$ *are given by the figure below:*



The four endpoints are roots of $\mathrm{Disc}_t(z^2t^2 + (z^2 - 2z + i)t + 1) = (z^2 + i)(z^2 - 4z + i)$*.*

We note that in the previous two examples, the endpoints of the curves containing the roots of $H_m(z)$ are roots of $\mathrm{Disc}_t D(z,t)$ and the roots of $H_m(z)$ are denser around these endpoints. In these particular examples, the reason for the first conclusion is mentioned in the first example and the reason for the latter comes from the fact that $q^{-1} + q + 1$ is denser around the endpoints when q is a m-th root of unity and the map $B(z)^2/A(z)C(z)$ does not affect such distribution. In general, these two conclusions depend on the map $B^2(z)/A(z)C(z)$. We give an example in which neither of these conclusions holds. Suppose $D(z,t) = t^2 + t + z + 1$. The roots of $H_{100}(z)$ are given in the figure below:



By complex expansion, we notice that

$$\Im\frac{B^2(z)}{A(z)C(z)} = -\frac{y}{1+2x+x^2+y^2},$$
$$\Re\frac{B^2(z)}{A(z)C(z)} = \frac{1+x}{1+2x+x^2+y^2}.$$

Theorem 6.1 implies the roots of $H_m(z)$ lie on the real interval $[-3/4,\infty)$. However these roots are not denser toward ∞, and ∞ is not a root of $\mathrm{Disc}_t D(z,t)$.

Now, what happens to Theorem 6.1 when the numerator is not 1? The proof of this theorem does not apply to general cases but it holds for some special generating functions. We provide another proof of Proposition 2.2. The sequence of polynomials $L((1+z)^m)$ has the generating function

$$\frac{1-(z+1)t}{(t-1)^2+t^2z^2}.$$

Partial fractions with $t_1=(1+iz)^{-1}$ and $t_2=(1-iz)^{-1}$ imply that the roots of $L((1+z)^m)$ satisfy

$$\frac{1-(z+1)t_1}{1-(z+1)t_2}=q^m$$

where $q=t_1/t_2$. We note that if the numerator is 1, this equation will be $q^m=1$. Thus q will lie on the unit circle and Theorem 6.1 holds. In this case with $t_1=(1+iz)^{-1}$ and $t_2=(1-iz)^{-1}$, this equation is $q^m=-i$. Thus q still lies on the unit circle and the conclusion of Theorem 6.1 is valid for this sequence of polynomials. The condition

$$\Im \frac{B^2(z)}{A(z)C(z)} = 0$$

implies the roots of $L((1+z)^m)$ are real or purely imaginary. We exclude the later case since if z is purely imaginary then $\Re(L(1+z)^m) \geq 1$ by the definition of L.

Next, we will show that in the case $D(z,t) = A(z)t^3 + B(z)t + 1$, the roots of $H_m(z)$ lie on a fixed algebraic curve. As in the proof of Theorem 6.1, we first consider the distribution of the quotient of roots $q = t_i/t_j$ of $D(z,t)$. Then we examine the root distribution of $H_m(z)$ using q-discriminants. This quotient lied on the unit circle in the previous section. In this section, we show that this quotient lies on the curve in Figure 6.1.

Lemma 6.2. *Suppose $\zeta_1, \zeta_2 \neq 0$ are complex numbers such that $1/\zeta_1 + 1/\zeta_2 + 1 = 0$ and*

$$\frac{\zeta_1^{m+1} - 1}{\zeta_1 - 1} = \frac{\zeta_2^{m+1} - 1}{\zeta_2 - 1}. \tag{6.4}$$

Then the set of solutions ζ_1 and ζ_2 is dense on the union $C_1 \cup C_2 \cup C_3$ as $m \to \infty$where the Cartesian equations of C_1, C_2 and C_3 are given by

$$\begin{aligned} C_1 &: (x+1)^2 + y^2 = 1, x \leq -\frac{1}{2}, \\ C_2 &: x = -\frac{1}{2}, -\frac{\sqrt{3}}{2} \leq y \leq \frac{\sqrt{3}}{2}, \\ C_3 &: x^2 + y^2 = 1, x \geq -\frac{1}{2}. \end{aligned}$$

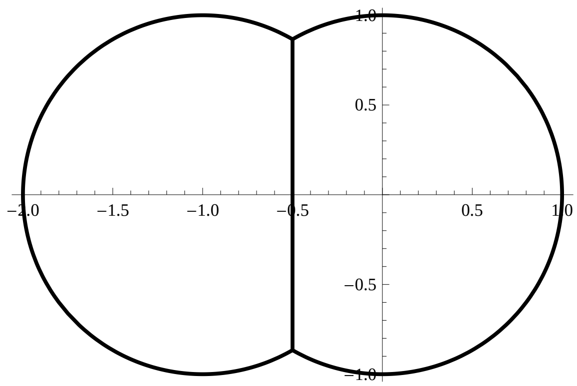


Figure 6.1: Distribution of the quotients of the roots of the cubic denominator

Proof. We note that there are at most $2m-2$ solutions $\zeta = \zeta_1 \neq 0, -2$ counting multiplicity. Let $m = 3n + k$ where $k = 1, 2, 3$. From implicit differentiation, we can check that the equation (6.4) has roots at $e^{2\pi i/3}, e^{2\pi i/3}$ with multiplicity $k-1$. After subtracting this number of roots from $2m-2$, we conclude that there are at most $6n$ roots $\zeta \neq 0, -2, e^{2\pi i/3}, e^{4\pi i/3}$. We first show that if $\zeta \neq -2$ is a root, then so is $-\zeta - 1$. From the two equations in the hypothesis, we note that $\zeta \neq 0, -1$ and

$$\sum_{k=0}^{m} \zeta^k = \sum_{k=0}^{m} \left(-\frac{\zeta}{\zeta+1} \right)^k .$$

Subtracting 1 then dividing by ζ and multiplying by $(\zeta+1)^m$ to both sides, we obtain

$$\begin{aligned} \sum_{k=0}^{m-1} \zeta^k (\zeta+1)^m + \sum_{k=0}^{m-1} (\zeta+1)^{m-k-1}(-\zeta)^k &= \sum_{k=0}^{m-1} \zeta^k (\zeta+1)^{m-k-1} \left((\zeta+1)^{k+1} - (-1)^{k+1} \right) \\ &= (\zeta+2) \sum_{k=0}^{m-1} \zeta^k (\zeta+1)^{m-k-1} \sum_{i=0}^{k} (\zeta+1)^{k-i}(-1)^i \\ &= (\zeta+2) \sum_{k=0}^{m-1} \sum_{i=0}^{k} \zeta^k (-\zeta-1)^{n-1-i}. \end{aligned}$$

By interchanging the summation and reversing the index of summation we obtain

$$\begin{aligned}\sum_{k=0}^{m-1}\sum_{i=0}^{k}\zeta^k(-\zeta-1)^{n-1-i} &= \sum_{i=0}^{m-1}\sum_{k=i}^{m-1}\zeta^k(-\zeta-1)^{m-1-i}\\ &= \sum_{i=0}^{m-1}\sum_{k=0}^{i}\zeta^{m-1-k}(-\zeta-1)^{i}.\end{aligned}$$

Hence we have the symmetry between ζ and $-1-\zeta$ in the double summation.

Our goal is to show that the number of roots $\zeta \neq 0, -2, e^{2\pi i/3}, e^{4\pi i/3}$ on $C_1 \cup C_2 \cup C_3$ counting multiplicity is at least $6n$. Then all roots will lie on $C_1 \cup C_2 \cup C_3$ since we have at most $6n$ roots $\zeta \neq 0, -2, e^{2\pi i/3}, e^{4\pi i/3}$. By the symmetry of roots mentioned above, if $\zeta \neq -2$ is a solution in C_1 then $(-1-1/\zeta, -\zeta-1)$ is a solution in $C_2 \times C_3$. Hence there is a bijection between roots in $C_1\backslash\{-2\}$, C_2 and C_3. Thus if $C_1\backslash\{e^{2\pi i/3}, e^{4\pi i/3}\}$ contains at least $2n+1$ roots then all of the roots lie on $C_1 \cup C_2 \cup C_3$. Let $\zeta = \zeta_1$ be a root on $C_1\backslash\{e^{2\pi i/3}, e^{4\pi i/3}\}$. The equation $1/\zeta_1 + 1/\zeta_2 + 1 = 0$ gives $\zeta_2 = \bar{\zeta}$. Thus

$$\Im\frac{\zeta^{m+1}-1}{\zeta-1} = 0.$$

Write $\zeta = re^{i\theta}$ where $r = -2\cos\theta$, $\cos\theta \leq -1/2$. Then complex expansion yields

$$r^{m+2}\sin m\theta - r^{m+1}\sin(m+1)\theta + r\sin\theta = 0.$$

Replace r by $-2\cos\theta$ and combine the first two terms to obtain

$$(-1)^{m+1}2^m\cos^m\theta\,(2\sin(m+1)\theta + \sin(m-1)\theta) + \sin\theta = 0.$$

We note that the left side has different signs if $\sin(m+1)\theta = 1$ and $\sin(m+1)\theta = -1$. Thus we can apply the Intermediate Value Theorem on several intervals whose boundaries are the solutions of $\sin(m+1)\theta = \pm 1$. The equations $\sin(m+1)\theta = \pm 1$ give

$$(m+1)\theta = \pm\frac{\pi}{2} + 2j\pi.$$

The condition $2\pi/3 < \theta < 4\pi/3$ and the fact that $m = 3n+k$, $k = 1, 2, 3$, yield

$$n + \frac{k+1}{3} \pm \frac{1}{4} < j < 2n + \frac{2(k+1)}{3} \pm \frac{1}{4}.$$

If $k = 1$, we have at least $2n+1$ roots coming from $2n+1$ intervals formed by $2n+2$ points

$$\frac{2j\pi \pm \pi/2}{m+1},$$

where $n < j \leq 2n+1$. If $k = 2$, we have at least $2n+1$ roots coming from $2n+1$ intervals formed by $2n+2$ points

$$\left\{\frac{2j - \pi/2}{m+1} : n+1 \leq j < 2n+2\right\} \cup \left\{\frac{2j + \pi/2}{m+1} : n+1 < j \leq 2n+2\right\}.$$

If $k = 3$, we have at least $2n+1$ roots coming from $2n+1$ intervals formed by $2n+2$ points

$$\frac{2j\pi \pm \pi/2}{m+1}$$

where $n+1 < j < 2n+2$. The density follows from the distribution of $2n+1$ roots mentioned above. The lemma follows.

$\square$

Theorem 6.3. *Let*

$$\frac{1}{A(z)t^3 + B(z)t + 1} = \sum_{m=0}^{\infty} H_m(z)t^m.$$

The roots of $H_m(z)$ which satisfy $A(z) \neq 0$, lie and are dense as $m \to \infty$ on a fixed curve

$\mathcal{C}_3$ given by

$$\Im \frac{\mathrm{Disc}_t(A(z)t^3 + B(z)t + 1)}{A^2(z)} = 0$$

and limited by

$$-27 \le \Re \frac{\mathrm{Disc}_t(A(z)t^3 + B(z)t + 1)}{A^2(z)} \le 0.$$

Proof. Suppose z is a root of $H_{m-1}(z)$ which satisfies $A(z) \neq 0$. It suffices to consider $\mathrm{Disc}_t(A(z)t^3 + B(z)t + 1) \neq 0$. Let t_1, t_2 and t_3 be the roots of $A(z)t^3 + B(z)t + 1$. By partial fractions, the generating function is

$$\frac{1}{A(z)(t_1 - t_2)(t_1 - t_3)(t - t_1)} + \frac{1}{A(z)(t_2 - t_1)(t_2 - t_3)(t - t_2)} + \frac{1}{A(z)(t_3 - t_1)(t_3 - t_2)(t - t_3)}$$

which can be written as

$$\sum_{m=1}^{\infty} \frac{t_1^{m+1}t_2^m - t_1^m t_2^{m+1} - t_1^{m+1}t_3^m + t_2^{m+1}t_3^m + t_1^m t_3^{m+1} - t_2^m t_3^{m+1}}{A(z)t_1^m t_2^m t_3^m (t_1 - t_2)(t_1 - t_3)(t_2 - t_3)} t^{m-1}.$$

Since z is a root of $H_{m-1}(z)$, we have

$$t_1^{m+1}t_2^m - t_1^m t_2^{m+1} - t_1^{m+1}t_3^m + t_2^{m+1}t_3^m + t_1^m t_3^{m+1} - t_2^m t_3^{m+1} = 0.$$

Dividing the equation by t_3^{2m+1} and letting $q = q_1 = t_1/t_3$, $q_2 = t_2/t_3$ we obtain

$$q_1^{m+1}q_2^m - q_1^m q_2^{m+1} - q_1^{m+1} + q_2^{m+1} + q_1^m - q_2^m = 0$$

where $q_1 + q_2 + 1 = 0$. The equation can be written as

$$q_1^m q_2^m (q_1 - q_2) - q_1^m (q_1 - 1) + q_2^m (q_2 - 1) = 0.$$

Since $q_1^m q_2^m(q_1 - q_2) = q_1^m q_2^m(q_1 - 1) - q_1^m q_2^m(q_2 - 1)$ and $q_1, q_2 \neq 0, 1$, this equation becomes

$$\frac{q_1^m - 1}{q_1^m(q_1 - 1)} = \frac{q_2^m - 1}{q_2^m(q_2 - 1)}.$$

Let $\zeta_1 = 1/q_1$ and $\zeta_2 = 1/q_2$ and add 1 to both sides. Then

$$\frac{\zeta_1^{m+1} - 1}{\zeta_1 - 1} = \frac{\zeta_2^{m+1} - 1}{\zeta_2 - 1}.$$

Thus ζ_1 and ζ_2 (and also q_1 and q_2) lie on the curve given in Lemma 6.2. Since q_1 and q_2 are given by a quotient of two roots, they are roots of the q-discriminant given by

$$\mathrm{Disc}_t(A(z)t^3 + B(z)t + 1; q) = -B^3(z)A(z)q^2(1+q)^2 - A^2(z)(1+q+q^2)^3.$$

This gives

$$\frac{B^3(z)}{A(z)} = -\frac{(1+q+q^2)^3}{q^2(1+q)^2}.$$

Since $\mathrm{Disc}_t(A(z)t^3 + B(z)t + 1)) = -4A(z)B^3(z) - 27A^2(z)$, it remains to show that the map

$$f(q) = -\frac{(1+q+q^2)^3}{q^2(1+q)^2}$$

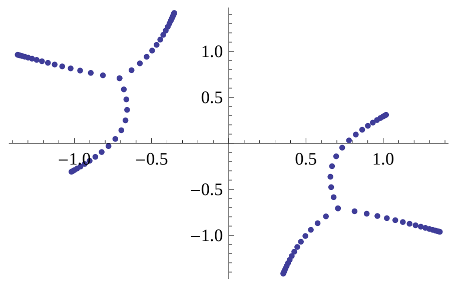
maps the curve in Figure 6.1 to the real interval $[-27/4, 0]$. Let q be a point on the this curve. We note that

$$f(q) = f(-1-q) = -\frac{(q^{-1}+1+q)^3}{q^{-1}+2+q}.$$

Since q lies on the curve in Figure 6.1, we have three possible cases $\bar{q} = -1 - q$, $|q| = 1$ or $|-1-q| = 1$. From the equation above, $\Im f(q) = 0$ in all of these cases. Furthermore, $f(q)$ obtains maximum and minimum when $q = 1$ and $q = -1/2$ respectively. The density of the roots of $H_m(z)$ follows from similar arguments in the proof of Theorem (6.1). Hence, the theorem follows.

□

Example. *We consider the case* $D(z,t) = t^3 + (z^2+i)t + 1$. *The roots of* $H_{50}(z)$ *are given by the figure below:*



By complex expansion, we have

$$\begin{aligned}\Im \frac{B^3(z)}{A(z)} &= (1+2xy)(-1+3x^4-4xy-10x^2y^2+3y^4),\\ \Re \frac{B^3(z)}{A(z)} &= (x-y)(x+y)(-3+x^4-12xy-14x^2y^2+y^4).\end{aligned}$$

Thus the roots of $H_m(z)$ *lie on parts of the curve*

$$(1+2xy)(-1+3x^4-4xy-10x^2y^2+3y^4) = 0.$$

Next, we will show that in the case $D(z,t) = A(z)t^4 + B(z)t + 1$ the roots of $H_n(z)$ lie on a fixed algebraic curve. As in the approach to Theorem 6.1 and Theorem 6.3, we first consider the distribution of the quotients of roots of $D(z,t)$.

Lemma 6.4. *Suppose* z *is a root of* $H_m(z)$ *and* $q = q(z)$ *is a quotient of two roots in* t *of* $A(z)t^4 + B(z)t + 1$. *Then the set of all such quotients belongs to and is dense as* $m \to \infty$ *on the following curve:*

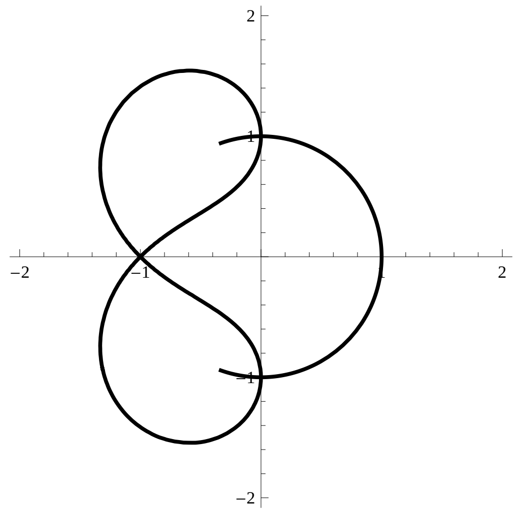


Figure 6.2: Distribution of the quotients of the roots of the quartic denominator

where the Cartesian equation of the quartic curve on the left is

$$1 + 2x + 2x^2 + 2x^3 + x^4 - 2y^2 + 2xy^2 + 2x^2y^2 + y^4 = 0,$$

and the curve on the right is the unit circle with real part at least $-1/3$.

Proof. Let t_1, t_2, t_3, and t_4 be the roots of the denominator $A(z)t^4 + B(z)t + 1$. By partial fractions, we have

$$\begin{aligned}\frac{1}{A(z)t^4 + B(z)t + C(z)} &= \frac{1}{A(z)(t-t_1)(t-t_2)(t-t_3)(1-t_4)} \\ &= \sum_{m=0}^{\infty} H_{m+1}(t_1, t_2, t_3, t_4)t^m,\end{aligned}$$

where

$$\begin{aligned}A(z)H_m(t_1,t_2,t_3,t_4) &= \frac{1}{t_1^m(t_1-t_2)(t_1-t_3)(t_1-t_4)} + \frac{1}{t_2^m(t_2-t_1)(t_2-t_3)(t_2-t_4)} \\ &\quad + \frac{1}{t_3^m(t_3-t_1)(t_3-t_2)(t_3-t_4)} + \frac{1}{t_4^m(t_4-t_1)(t_4-t_2)(t_4-t_3)}.\end{aligned}$$

Let $q_1 = t_1/t_4$, $q_2 = t_2/t_4$, $q_3 = t_3/t_4$. We put all terms over a common denominator and then divide the numerator by t_4^{3m}. The condition $H_{m-1}(t_1, t_2, t_3, t_4) = 0$ implies

$$\begin{aligned} 0 \;=\; & q_1^{m+1}(-q_2^{m-1}q_3^{m-1}(q_2 - q_3) + q_2^m - q_3^m - q_2^{m-1} + q_3^{m-1}) \\ & + q_1^m(q_2^{m-1}q_3^{m-1}(q_2^2 - q_3^2) - q_2^{m+1} + q_3^{m+1} + q_2^{m-1} - q_3^{m-1}) \\ & + q_1^{m-1}(-q_2^m q_3^m(q_2 - q_3) + q_2^{m+1} - q_3^{m+1} - q_2^m + q_3^m) \\ & + q_2^{m-1}q_3^{m-1}(q_2 - q_3) - q_2^{m-1}q_3^{m-1}(q_2^2 - q_3^2) + q_2^m q_3^m(q_2 - q_3). \end{aligned} \tag{6.5}$$

From the fact that

$$\begin{aligned} t_1 + t_2 + t_3 + t_4 &= 0 \\ t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4 &= 0, \end{aligned}$$

we have

$$\begin{aligned} q_2 + q_3 &= -1 - q_1 \\ q_2q_3 &= q_1^2 + q_1 + 1. \end{aligned}$$

From the symmetric reductions, the right side of (6.5), after divided by $q_2 - q_3$, is a polynomial in q_1 of degree $3m - 1$. We leave it to a computer algebra system to check for the root distribution of this polynomial in the case $m \leq 5$. We now assume that $m \geq 6$. We will show that the number of roots q_1 lying on the two curves in Figure 6.2 is at least $3m - 1$. The first step is to show that if the set of q_1 belongs to and is dense as $m \to \infty$ on the unit circle with $\Re q_1 \geq -1/3$ then the set of q_2 and q_3 belongs to and is dense as $m \to \infty$ on the quartic curve given in the lemma. Then we will find the number of roots q_1 on the unit circle with $\Re q_1 \geq -1/3$.

Suppose $q_1 = e^{i\pi\theta}$ lies on the unit circle and $1 \geq \cos\theta \geq -1/3$. We note that q_2 and q_3

are the two roots of the equation

$$f(q) := q^2 + (1 + q_1)q + q_1^2 + q_1 + 1 = 0.$$

Thus

$$q = \frac{-1 - e^{i\theta} \pm i e^{i\theta/2}\sqrt{6\cos\theta + 2}}{2}.$$

After splitting the real and imaginary parts, and check these parts with the equation

$$1 + 2x + 2x^2 + 2x^3 + x^4 - 2y^2 + 2xy^2 + 2x^2y^2 + y^4 = 0,$$

the first step follows.

We now compute the number of roots $q_1 = e^{i\pi\theta}$ with $\cos\theta \geq -1/3$. We first consider $q_1 \neq \pm i, 1$. Let

$$2\zeta = \left(\frac{q_2}{q_3}\right)^{1/2} + \left(\frac{q_3}{q_2}\right)^{1/2}.$$

From the definition of the Chebyshev polynomial of the second kind as the quotient of two trigonometric functions (see [11]), we have

$$\begin{aligned} \frac{q_2^m - q_3^m}{q_2 - q_3} &= (q_2q_3)^{(m-1)/2}\frac{(q_2/q_3)^{m/2} - (q_3/q_2)^{m/2}}{(q_2/q_3)^{1/2} - (q_3/q_2)^{1/2}} \\ &= (q_2q_3)^{(m-1)/2}U_{m-1}(\zeta) \end{aligned}$$

where

$$\begin{aligned} \zeta^2 &= \frac{(q_1+1)^2}{4(q_1^2 + q_1 + 1)} \\ &= \frac{1}{4(2\cos\theta + 1)} + \frac{1}{4} \in \mathbb{R}. \end{aligned}$$

We divide (6.5) by $q_2 - q_3$ and rewrite it in terms of Chebyshev polynomials

$$\begin{aligned} 0 &= U_m(\zeta)(-q_1^m + q_1^{m-1})(q_1^2 + q_1 + 1)^{m/2} \\ &\quad + U_{m-1}(\zeta)(q_1^{m+1} - q_1^{m-1})(q_1^2 + q_1 + 1)^{(m-1)/2} \\ &\quad + U_{m-2}(\zeta)(-q_1^{m+1} + q_1^m)(q_1^2 + q_1 + 1)^{(m-2)/2} \\ &\quad + (q_1^2 + q_1 + 1)^{m-1}(-3q_1^{m+1} - 2q_1^m - q_1^{m-1} + q_1^2 + 2q_1 + 3). \end{aligned}$$

We divide this equation by $q_1^{(3m-2)/2}(1 - q_1)(q_1 + q_1^{-1} + 1)^{m/2}$ and write $(q_1^m - 1)/(q_1 - 1)$ in terms of Chebyshev polynomials. We obtain

$$\begin{aligned} 0 &= U_m(\zeta) + 2\zeta U_{m-1}(\zeta) + U_{m-2}(\zeta)/(q_1 + q_1^{-1} + 1) \\ &\quad + (q_1 + q_1^{-1} + 1)^{m/2-1} \left(3U_m(\xi) + 2U_{m-2}(\xi) + U_{m-4}(\xi)\right), \end{aligned}$$

where

$$\begin{aligned} \xi^2 &= \frac{(q_1 + 1)^2}{4q_1} \\ &= \frac{2\cos\theta + 2}{4}. \end{aligned}$$

Finally, we replace $1/(q_1 + q_1^{-1} + 1)$ by $(4\zeta^2 - 1)$ and use the recurrence definition of the Chebyshev polynomials to rewrite this equation in the symmetric form below:

$$\begin{aligned} 0 &= (4\zeta^2 - 1)^{(m-2)/4} \left(3U_m(\zeta) + 2U_{m-2}(\zeta) + U_{m-4}(\zeta)\right) \\ &\quad + (4\xi^2 - 1)^{(m-2)/4} \left(3U_m(\xi) + 2U_{m-2}(\xi) + U_{m-4}(\xi)\right). \end{aligned} \tag{6.6}$$

From this symmetric form, the right expression remains the same if we interchange ζ and ξ or if we interchange $\cos\theta$ and $-\cos\theta/(2\cos\theta + 1)$. Thus the numbers of roots q_1 are the same in the two cases $0 < \cos\theta < 1$ and $-1/3 < \cos\theta < 0$. It is sufficient to count the number of roots $0 < \cos\theta < 1$ or $1/2 < \xi^2 < 1$. Let $\cos\alpha = \xi$ and $U_m(\xi) = \sin(m+1)\alpha/\sin\alpha$ where $-\pi/4 < \alpha < \pi/4$, $\alpha \neq 0$. The idea is to show that in this case the summand

$$(4\xi^2 - 1)^{(m-2)/4}\left(3U_m(\xi) + 2U_{m-2}(\xi) + U_{m-4}(\xi)\right)$$

dominates the right expression of (6.6). Thus we can apply the Intermediate Value Theorem since this summand has different signs when $\sin(m+1)\alpha = 1$ and when $\sin(m+1)\alpha = -1$. Suppose $\sin(m+1)\alpha = \pm 1$ and $-\pi/4 < \alpha < \pi/4$. We will show

$$\left|(4\xi^2 - 1)^{(m-2)/2}\left(3U_m(\xi) + 2U_{m-2}(\xi) + U_{m-4}(\xi)\right)\right| \geq \left|3U_m(\zeta) + 2U_{m-2}(\zeta) + U_{m-4}(\zeta)\right|. \tag{6.7}$$

Let $\zeta = \cos\beta$. Using the fact that $1/3 < \zeta^2 < 1/2$ and $U_m(\zeta) = \sin(m+1)\beta/\sin\beta$, we obtain the following upper bound for the right hand side of (6.7):

$$\left|3U_m(\zeta) + 2U_{m-2}(\zeta) + U_{m-4}(\zeta)\right| \leq 6\sqrt{2}.$$

Since

$$\alpha = \frac{\pi}{4}\frac{(4k \pm 2)}{m+1} \tag{6.8}$$

where $k \in \mathbb{Z}$ and $-\pi/4 < \alpha < \pi/4$, we have

$$|\alpha| \leq \frac{\pi}{4}\left(1 - \frac{1}{m+1}\right).$$

Thus

$$\cos\alpha \geq \frac{\sqrt{2}}{2}\left(\cos\frac{\pi}{4(m+1)} + \sin\frac{\pi}{4(m+1)}\right).$$

This inequality gives

$$2\cos\theta = 4\cos^2\alpha - 2$$

$$\geq \quad 4\sin\frac{\pi}{2(m+1)}.$$

From the definition of the Chebyshev polynomial, we have

$$\begin{aligned}
U_{m-1}(\xi) &= \frac{\sin m\alpha}{\sin\alpha} \\
&= \frac{\sin(m+1)\alpha\cos 2\alpha - \cos(m+1)\alpha\sin 2\alpha}{\sin\alpha} \\
&= \frac{\sin(m+1)\alpha\cos 2\alpha}{\sin\alpha}.
\end{aligned}$$

With similar computations for $U_{m-4}(\xi)$, we obtain

$$|3U_m(\xi) + 2U_{m-2}(\xi) + U_{m-4}(\xi)| = \frac{|\sin(m+1)\alpha||3 + 2\cos 2\alpha + \cos 4\alpha|}{|\sin\alpha|}.$$

Since $\sin(m+1)\alpha = \pm 1$ and $\cos 2\alpha \geq 0$, the right side is at least $2\sqrt{2}$. Thus we have

$$\begin{aligned}
\left|(2\cos\theta+1)^{(m-2)/2}\left(3U_m(\xi) + 2U_{m-2}(\xi) + U_{m-4}(\xi)\right)\right| &= \left(1 + 4\sin\frac{\pi}{2(m+1)}\right)^{(m-2)/2} 2\sqrt{2} \\
&\geq 6\sqrt{2}
\end{aligned}$$

when $m \geq 6$. The inequality (6.7) follows. By the Intermediate Value theorem, we have at least one root when $\sin(m+1)\alpha$ changes between -1 and 1 with $-\pi/4 < \alpha < \pi/4$. From the formula (6.8), the number of roots q_1 when $0 < \cos\theta < 1$ is at least $2(\lfloor (m-2)/4 \rfloor)$. By symmetry, the number of roots $q_1 \neq \pm i, 1$ with $\Re q_1 > -1/3$ on the unit circle is at least $4(\lfloor (m-2)/4 \rfloor)$. Note that each of these roots gives two more roots q_2 and q_3 on the quartic curve.

It remains to check the multiplicities of $q_1 = \pm i, 1$ in the equation (6.5). We note that this equation has a root $q_1 = 1$ with multiplicity at least 1. In the case $q_1 = 1$ we obtain four more roots q_2, q_2^{-1}, q_3, and q_3^{-1}. We now consider the case $q_1 = \pm i$. The equation

$q^2+(1+q_1)q+q_1^2+q_1+1=0$ where $q=q_2,q_3$ gives $(q_2,q_3)=(-1,i)$ or $(q_2,q_3)=(i,-1)$ when $q_1=i$ and $(q_2,q_3)=(-1,-i)$ or $(q_2,q_3)=(-i,-1)$ when $q_1=-i$. Hence each of the roots $q_1=\pm i$ gives us another root at -1 with the same multiplicity. To check the multiplicities at $q_1=\pm i$, we need to differentiate the equation (6.5) with respect to q_1. We obtain its derivatives by applying implicit differentiation to the equation $q^2+(1+q_1)q+q_1^2+q_1+1=0$. After substituting $q_1=\pm i$ in (6.5) and its derivatives, we see that the multiplicity of $\pm i$ is

$$\begin{cases} 2 & \text{if } m=4k \\ 3 & \text{if } m=4k+1 \\ 0 & \text{if } m=4k+2 \\ 1 & \text{if } m=4k+3 \end{cases}.$$

The table below tabulates the $3m-1$ roots of (6.5).

	$m=4k$	$m=4k+1$	$m=4k+2$	$m=4k+3$
$q_1=e^{i\theta}$,$\Re q_1>-1/3$, $q_1\neq\pm i,1$	$3(4k-4)$	$3(4k-4)$	$12k$	$12k$
$q_1=1$	5	5	5	5
$q_1=\pm i$	6	9	0	3
Total	$12k-1$	$12k+2$	$12k+5$	$12k+8$

All the roots counted on the table lie on the curves given in the lemma. Also, as a consequence of Intermediate Value Theorem applied to the intervals formed by $\sin(m+1)\alpha=\pm1$, the roots q_1 are dense on the portion of the unit circle with real part at least $-1/3$. The lemma follows.

$\square$

Theorem 6.5. *Let*

$$\frac{1}{A(z)t^4+B(z)t+1}=\sum_{m=0}^{\infty}H_m(z)t^m.$$

The roots of $H_m(z)$ which satisfy $A(z) \neq 0$ lie and are dense as $m \to \infty$ on a fixed curve $\mathcal{C}_4$ given by

$$\Im \frac{\mathrm{Disc}_t(A(z)t^4 + B(z)t + 1)}{A^3(z)} = 0$$

and limited by

$$0 \leq \Re \frac{\mathrm{Disc}_t(A(z)t^4 + B(z)t + 1)}{A^3(z)} \leq 4^4.$$

Proof. From the definition of q-discriminant in (1.2), we have

$$\mathrm{Disc}_t(A(z)t^4 + B(z)t + 1; q) = -A^2(z)B^4(z)q^3(1 + q + q^2)^3 + A^3(z)(1 + q + q^2 + q^3)^4.$$

If q is a quotient of the two roots of $A(z)t^4 + B(z)t + 1$, then

$$\frac{B^4(z)}{A(z)} = \frac{(1 + q + q^2 + q^3)^4}{q^3(1 + q + q^2)^3}.$$

Let $f(q)$ be the function on the right side. We note that $f(q)$ maps $q_1 = e^{i\theta}$ with $\Re q_1 \geq -1/3$ to the real interval $[0, 4^4/3^3]$ since

$$f(q_1) = \frac{(q_1^{3/2} + q_1^{-3/2} + q_1^{1/2} + q_1^{-1/2})^4}{(q_1 + q_1^{-1} + 1)^3}.$$

If q is a point on the quartic curve in Lemma 6.4 then q and q_1 are related by

$$q_1^2 + q^2 + q_1 q + q_1 + q + 1 = 0.$$

Multiplying this equation by $q_1 - q$, we obtain

$$q_1^3 + q_1^2 + q_1 = q^3 + q^2 + q.$$

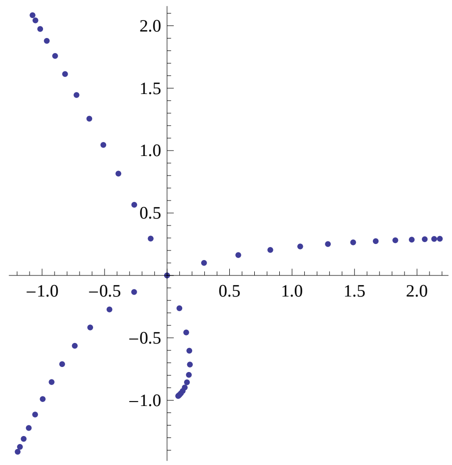
Thus by the definition of $f(q)$, we have $f(q) = f(q_1)$. Since

$$\mathrm{Disc}_t(A(z)t^4 + B(z)t + 1) = -3^3 A^2(z)B^4(z) + 4^4 A^3(z),$$

the roots of $H_m(z)$ lie on the curve $\mathcal{C}_4$. The density of these roots follows from similar arguments in the proof of Theorem (6.1).

$\square$

Example. *Let $D(z,t) = (z+i)t^4 + zt + 1$. The roots of $H_{50}(z)$ are given in the figure below.*



By complex expansion, Theorem 6.5 implies that the roots of $H_m(z)$ lie on parts of the curve

$$-x^4 + 3x^4y + 6x^2y^2 + 2x^2y^3 - y^4 - y^5 = 0.$$

Remark. *One may try to find the root distribution of $H_n(z)$ in the case $D(z,t) = A(z)t^5 + B(z)t + 1$. From computer experiments, the distribution of the quotients of roots of $D(z,t)$ in the case $m = 20$ is given in the figure below.*

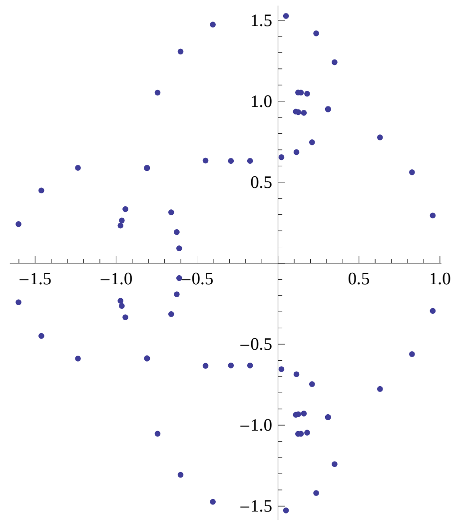


Figure 6.3: Distribution of the quotients of the roots of the quintic denominator

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